Variational principles for set-valued mappings
with applications to multiobjective optimization

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Abstract: This paper primarily concerns the study of general
classes of constrained multiobjective optimization problems (includ-
ing those described via set-valued and vector-valued cost mapp-
ings) from the viewpoint of modern variational analysis and gener-
alized differentiation. To proceed, we first establish two variational
principles for set-valued mappings, which—being certainly of inde-
pendent interest—are mainly motivated by applications to multiobjec-
tive optimization problems considered in this paper. The first variati-
onal principle is a set-valued counterpart of the seminal deriva-
tive-free Ekeland variational principle, while the second one is a set-valued ex-
tension of the subdifferential principle by Mordukhovich and Wang,
formulated via an appropriate subdifferential notion for set-valued
mappings with values in partially ordered spaces. Based on these
variational principles and corresponding tools of generalized differen-
tiation, we derive new conditions of the coercivity and Palais-Smale
types ensuring the existence of optimal solutions to set-valued opti-
mization problems with noncompact feasible sets in infinite dimen-
sions and then obtain necessary optimality and suboptimality con-
ditions for nonsmooth multiobjective optimization problems with
general constraints, which are new in both finite-dimensional and
infinite-dimensional settings.

Keywords: multiobjective optimization, variational principles,
generalized differentiation, existence of optimal solutions, necessary
optimality and suboptimality conditions.

1. Introduction

The primary goal of this paper is to study constrained multiobjective optimization problems generally given by

\[ \text{minimize } F(x) \text{ subject to } x \in \Omega \subset X \]  

(1)
by using advanced tools of modern variational analysis and generalized differentiation. In (1), the cost mapping $F: X \rightrightarrows Z$ may be set-valued, and “minimization” is understood with respect to some partial ordering on $Z$. Thus (1) is a problem of set-valued optimization, while the term of vector optimization is usually used when $F = f: X \to Z$ is a single-valued mapping. In this paper we unify both set-valued and vector optimization problems under the name of multiobjective optimization.

There is an abundant literature on various problems of multiobjective optimization. One of the first work in modern variational theory for such problems was done by Rolewicz; see Rolewicz (1975) and Pallaschke and Rolewicz (1998, Chapter 10). We refer the reader to the books of Chen, Huang, Yang (2005), Göpfert et al. (2003), Jahn (2004), Luc (1989), Mordukhovich (2006b), Pallaschke and Rolewicz (1998) and the bibliographies therein for more information on history, results, and methods in multiobjective optimization and related problems.

A characteristic feature of the current stage of variational analysis is the broad usage of modern variational principles, started with the seminal work by Ekeland (1974). The fundamental Ekeland variational principle asserts that, given a proper and lower semicontinuous function $\varphi: X \to \overline{\mathbb{R}} := (-\infty, \infty]$ bounded from below on the complete metric space $(X, d)$, for every $\varepsilon > 0$, $\lambda > 0$, and $x_0 \in X$ with $\varphi(x_0) < \inf_X \varphi(x) + \varepsilon$ there is $\bar{x} \in X$ satisfying the conditions $\varphi(\bar{x}) \leq \varphi(x_0)$, $d(\bar{x}, x_0) \leq \lambda$, and

$$\varphi(x) - \varphi(\bar{x}) + \frac{\varepsilon}{\lambda} d(x, \bar{x}) > 0 \text{ whenever } x \in X \text{ with } x \neq \bar{x}.$$ (2)

Note that (2) means that the perturbed function $\varphi(x) + (\varepsilon/\lambda)d(x, \bar{x})$ attains its strict global minimum over $X$ at $\bar{x}$. If $X$ is Banach and $f$ is Gâteaux differentiable, then (2) easily implies the perturbed stationary condition

$$\|\nabla \varphi(\bar{x})\| \leq \frac{\varepsilon}{\lambda},$$ (3)

which can be treated as a suboptimality condition to the problem of minimizing $\varphi(x)$—with no assumption on the existence of optimal solutions to this problem over $X$ particularly restrictive in infinite dimensions—and which was among the strongest original motivations for developing Ekeland’s variational principle in Ekeland (1974) and its subsequent applications.

When $\varphi$ is nonsmooth—just extended-real-valued, lower semicontinuous, and bounded from below as in the afore-mentioned Ekeland general result—another variational principle is established by Mordukhovich and Wang (2002) under the name of subdifferential variational principle. It gives the same conclusions as Ekeland’s principle with replacing the minimization condition (2) by the subdifferential one:

$$\|x^*\| \leq \varepsilon/\lambda \text{ for some } x^* \in \tilde{\partial} \varphi(\bar{x}),$$ (4)
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where \( \hat{\partial} \varphi(\bar{x}) \) stands for the so-called Fréchet subdifferential of \( \varphi \) at \( \bar{x} \) defined by

\[
\hat{\partial} \varphi(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\},
\]

and where the space \( X \) is assumed to be Asplund, i.e., a Banach space whose separable subspaces have separable duals; see, e.g., Phelps (1993) for more information and references on the broad class of Asplund spaces that includes, in particular, all reflexive Banach spaces.

The subdifferential variational principle is established in Mordukhovich and Wang (2002) (see also Mordukhovich, 2006a, Theorem 2.28) as a consequence of (actually an equivalence to) the extremal principle, which is a variational counterpart of local separation for nonconvex sets being a variational principle of the geometric type independent of the analytic Ekeland variational principle; see the books Mordukhovich (2006 a,b) for a comprehensive variational theory and numerous applications of the extremal principle. Observe that the subdifferential condition (4) is a nonsmooth counterpart of the almost stationary condition (3); furthermore, it implies certain enhanced versions of (analytic) smooth variational principles under additional smoothness assumptions of the space \( X \) in question; see Mordukhovich (2006a, Subsection 2.3.3).

In this paper we derive an appropriate analog of the afore-mentioned subdifferential variational principle for the set-valued (in particular, vector-valued) mappings with values in partially ordered spaces. We need such a result for the subsequent applications to constrained multiobjective optimization problems of type (1). The proof of the set-valued subdifferential variational principle (SVSVP) obtained in this paper is based on the extremal principle and a new version of the set-valued Ekeland variational principle (SVEVP) established below. The required version of the latter needed for our purposes (while certainly of independent interest) is different from various vector and set-valued extensions of Ekeland’s seminal result known in the literature; see, e.g., Bao and Khanh (2003); Bianchi, Kassay and Pini (2005); Chen, Huang and Yang (2005); Göpfert et al. (2003); Ha (2003, 2005); Hamel (2005); Hamel and Löhne (2006); Khanh (1989) and the references therein as well as further comments in Section 3.

The rest of the paper is organized as follows. In Section 2 we briefly review (for the reader’s convenience) certain basic tools of variational analysis and generalized differentiation widely used in the paper. Then we introduce new subdifferential notions for set-valued mappings (in particular, for vector-valued mappings) with values in partially ordered spaces and establish some of their important properties needed in the sequel.

In Section 3 we first derive a new version of the SVEVP and then use it in the proof of the new SVSVP via the extremal principle. The formulation of the SVSVP result, which plays a crucial role in the subsequent applications in this paper, involves the subdifferentials of set-valued mappings introduced in
Section 2. We discuss relationships of the results obtained with those known in the literature.

Section 4 contains applications of the variational techniques and principles developed in Section 3 to deriving efficient conditions for the existence of optimal solutions to set-valued constrained optimization problems. In particular, we establish new conditions of the coercivity type and of the subdifferential Palais-Smale type for set-valued and nonsmooth single-valued mappings ensuring the existence of weak minimizers to the multiobjective optimization problems under consideration.

The concluding Section 5 is devoted to applications of the variational principles established in Section 3 and some basic calculus rules of generalized differentiation from Mordukhovich (2006a) to deriving necessary optimality conditions for multiobjective optimization problems with general geometric constraints as well as their specifications for multiobjective problems of mathematical programming with equality and inequality constraints given by nonsmooth functions. In this section we also obtain suboptimality conditions for the aforementioned multiobjective problems, which do not assume the existence of optimal solutions and are important for both theoretical and numerical aspects of multiobjective optimization.

Throughout the paper we use standard notation from variational analysis and set-valued optimization; see the books Jahn (2004), Mordukhovich (2006a), Rockafellar and Wets (1998). Some special symbols are described in the text when they are introduced. Recall that $\mathbb{N} = \{1, 2, \ldots\}$, that $\mathcal{B}$ and $\mathcal{B}^*$ stand for the closed unit balls of the space in question and its topologically dual, and that $x \xrightarrow{\mathcal{G}} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. Unless otherwise stated, the norm on the product $X \times Y$ of Banach spaces is defined by

$$
\| (x, y) \| := \| x \| + \| y \|, \quad (x, y) \in X \times Y.
$$

Given a set-valued mapping $G: X \rightrightarrows X^*$ between a Banach space $X$ and its dual space $X^*$, the Painlevé-Kuratowski sequential outer/upper limit of $F$ as $x \to \bar{x}$ is defined by

$$
\operatorname{Lim sup}_{x \to \bar{x}} G(x) = \left\{ x^* \in X^* \left| \exists \text{ sequences } x_k \to \bar{x}, \ x_k \xrightarrow{w^*} x^* \right. \right. \left. \left. \text{such that } x_k^* \in G(x_k) \text{ for all } k \in \mathbb{N} \right\},
$$

where $w^*$ signifies the weak* convergence on $X^*$.

2. Subdifferentials of set-valued mappings

The primary goal of this section is to introduce and discuss new notions of subdifferentials for set-valued and vector-valued mappings with values in partially ordered spaces. To proceed, we first need to recall some well-recognized generalized differential constructions of variational analysis widely used in this paper.
We mainly follow the recent books by Mordukhovich (2006a,b), where the reader can find more details, references, and discussions. We also recommend the book by Rockafellar and Wets (1998) for related and additional material in finite dimensions and the one by Borwein and Zhu (2005) for that in Fréchet smooth spaces.

Given a nonempty subset Ω ⊂ X of a Banach space X, define the collection of ε-normals to Ω at ȃ ∈ Ω by

\[ \hat{N}_\varepsilon(ȃ; \Omega) := \left\{ x^* \in X^* \mid \limsup_{\varepsilon \downarrow 0} \frac{\langle x^*, x - ȃ \rangle}{\|x - ȃ\|} \leq \varepsilon \right\}, \quad \varepsilon \geq 0, \]  

(7)

with \( \hat{N}_0(ȃ; \Omega) := \emptyset \) if ȃ \( \notin \Omega \). For \( \varepsilon = 0 \) in (7), the construction \( \hat{N}(ȃ; \Omega) := \hat{N}_0(ȃ; \Omega) \) is known as the Fréchet normal cone (or prenormal cone) to Ω at ȃ.

When \( X = \mathbb{R}^n \), the dual/polar cone to \( \hat{N}(ȃ; \Omega) \) agrees with the (Bouligand-Severi) contingent cone to Ω at ȃ. Note that the Fréchet subdifferential \( \partial \varphi(ȃ) \) defined in (5) for an extended-real-valued function \( \varphi: X \to \mathbb{R} \) finite at ȃ admits the following equivalent geometric representation:

\[ \partial \varphi(ȃ) = \left\{ x^* \in X^* \mid (x^*, -1) \in \hat{N}(\langle ȃ, \varphi(ȃ) \rangle; \text{epi} \varphi) \right\} \]  

(8)

via Fréchet normals to the epigraph \( \text{epi} \varphi := \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\} \). The basic (limiting, Mordukhovich) normal cone to Ω at ȃ is defined by

\[ N(ȃ; \Omega) := \limsup_{\varepsilon \downarrow 0} \hat{N}_\varepsilon(ȃ; \Omega) \]  

(9)

via the Painlevé-Kuratowski sequential outer limit (6). If the space X is Asplund and the set Ω is locally closed around ȃ, we can equivalently put \( \varepsilon = 0 \) in (9). Note that both cones \( \hat{N}(ȃ; \Omega) \) and \( N(ȃ; \Omega) \) reduce to the normal cone of convex analysis for convex sets Ω.

Having now a set-valued mapping \( F: X \rightrightarrows Z \) between Banach spaces with the graph

\[ \text{gph } F := \{(x, z) \in X \times Z \mid z \in F(x)\}, \]

define its ε-coderivative \( \hat{D}_F^*F(ȃ, \tilde{z}) : Z^* \rightrightarrows X^* \) at \((ȃ, \tilde{z}) \in \text{gph } F\) by

\[ \hat{D}_F^*F(ȃ, \tilde{z})(z^*) := \left\{ x^* \in X^* \mid (x^*, -z^*) \in \hat{N}_\varepsilon(\langle ȃ, \tilde{z} \rangle; \text{gph } F) \right\}, \quad \varepsilon \geq 0, \]  

(10)

where \( \hat{D}_F^*F(ȃ, \tilde{z})(z^*) = \emptyset \) for \((x, \tilde{y}) \notin \text{gph } F\), and where \( \hat{D}_F^*F(ȃ, \tilde{z}) := \hat{D}_F^*F(ȃ, \tilde{z}) \) is a positively homogeneous set-valued mapping called the Fréchet coderivative of \( F \) at \((ȃ, \tilde{z})\). Based on (10) considered at points nearby the reference one, construct as in Mordukhovich (2006a) two (sequential) limiting coderivatives of \( F \) at \((ȃ, \tilde{z})\) called, respectively, the normal coderivative

\[ D_N^*F(ȃ, \tilde{z})(z^*) := \limsup_{\varepsilon \downarrow 0} \hat{D}_F^*F(ȃ, \tilde{z})(z^*) \]  

(11)
and the mixed coderivative of $F$ at $(\bar{x}, \bar{z})$ that is given by

$$D^*_M F(\bar{x}, \bar{z})(\bar{z}^*) := \limsup_{(x,z) \to (\bar{x}, \bar{z}) \atop \varepsilon \downarrow 0} \hat{D}^*_F(x, z)(\varepsilon)\bar{z}^*,$$

We can equivalently put $\varepsilon = 0$ in (11) and (12) if both spaces $X$ and $Z$ are Asplund and if the mapping $F$ is locally closed-graph around $(\bar{x}, \bar{z})$.

Note that, by definition (6) of the sequential outer limit, the only difference between (11) and (12) is that the weak$^*$ convergence $w^*$ is used in (11) on both dual spaces $X^*$ and $Z^*$, while in (12) the strong/norm convergence is employed on $Z^*$ versus the weak$^*$ convergence on $X^*$. Thus these limiting coderivatives agree when $\dim Z < \infty$ (they both reduce to the original construction by Mordukhovich, 2006a, see also the references and commentaries therein), while $D^*_M F(\bar{x}, \bar{z})$ may be essentially smaller than $D^*_N F(\bar{x}, \bar{z})$ even for single-valued Lipschitzian mappings $f: \mathbb{R} \to H$ to an arbitrary Hilbert space $H$ as in Mordukhovich (2006a, Example 1.35). Note that the normal coderivative (11) can be equivalently defined by

$$D^*_N F(\bar{x}, \bar{z})(\bar{z}^*) = \{x^* \in X^* | (x^*, -\bar{z}^*) \in N((\bar{x}, \bar{z}); \text{gph} F)\}$$

via the basic normal cone (9) to the graph of $F$.

Now let us consider a set-valued mapping $F: X \rightrightarrows Z$ between Banach spaces, where $Z$ is partially ordered by a convex and closed cone $\Theta \subset Z$. Denoting the ordering relation on $Z$ under consideration by “$\leq$”, we therefore have its description:

$$z_1 \leq z_2 \iff z_2 - z_1 \in \Theta.$$  

(13)

Given $F: X \rightrightarrows Z$, define its (generalized) epigraph with respect to the above order by

$$\text{epi} F := \{(x, z) \in X \times Z | z \in F(x) + \Theta\}$$

and associate with $F$ the epigraphical multifunction $\mathcal{E}_F: X \rightrightarrows Z$ defined by

$$\mathcal{E}_F(x) := \{z \in Z | z \in F(x) + \Theta\} \quad \text{with} \quad \text{gph} \mathcal{E}_F = \text{epi} F,$$

(14)

where the ordering cone $\Theta$ is not mentioned in the epigraphical notation for simplicity.

Our goal is to introduce appropriate subdifferentials of set-valued mappings with values in partially ordered spaces by using the corresponding coderivatives of the associated epigraphical multifunctions. Although there are many various definitions of subdifferentials for (single-valued) vector functions with values in partially ordered spaces, our coderivative approach and the subdifferential constructions below are different from those known in the literature (see, e.g., a very good survey on vector subdifferentials by Stamate, 2003). Furthermore,
our constructions apply to set-valued mappings/multifunctions with values in partially ordered spaces, which is important for the main results of this paper.

The following definition contains only those subdifferential constructions, which are used in this paper. Based on the coderivative approach and employing various limiting procedures on dual spaces, the reader can construct other subdifferentials that may be different from the ones given below in infinite dimensions.

**Definition 1** (subdifferentials of set-valued mappings) Let $F: X \rightrightarrows Z$ be a mapping between Banach spaces, let $\Theta \subset Z$ be a convex and closed cone that generates a partial order on $Z$ by (13), and let $(\bar{x}, \bar{z}) \in \text{epi} F$. We define the following subdifferentials of $F$ at $(\bar{x}, \bar{z})$ via the corresponding coderivatives of the epigraphical multifunction (14):

- **the $\varepsilon$-subdifferential of $F$ at $(\bar{x}, \bar{z})$** by

  $$\hat{\partial}_\varepsilon F(\bar{x}, \bar{z}) := \{ x^* \in X^* \mid x^* \in \hat{D}^*_\varepsilon E_F(\bar{x}, \bar{z})(z^*), -z^* \in N(0; \Theta), \|z^*\| = 1 \},$$

  $$\varepsilon \geq 0,$$  

  (15)

  where $\hat{\partial}F(\bar{x}, \bar{z}) := \hat{\partial}_0 F(\bar{x}, \bar{z})$ is the Fréchet subdifferential of $F$ at this point;

- **the normal subdifferential of $F$ at $(\bar{x}, \bar{z})$** by

  $$\partial N F(\bar{x}, \bar{z}) := \{ x^* \in X^* \mid x^* \in D^*_N E_F(\bar{x}, \bar{z})(z^*), -z^* \in N(0; \Theta), \|z^*\| = 1 \};$$

  (16)

- **the singular subdifferentials of $F$ at $(\bar{x}, \bar{z})$** by

  $$\partial^\infty F(\bar{x}, \bar{z}) := D^*_M E_F(\bar{x}, \bar{z})(0).$$

  (17)

As usual, we drop $\bar{z} = f(\bar{x})$ in the subdifferential notation (15)–(17) if $F = f: X \to Z$ is single-valued. When $\varphi: X \to \overline{\mathbb{R}}$ is an extended-real-valued function finite at $\bar{x}$ with the standard order $\Theta = \mathbb{R}_+$ on $\mathbb{R}$, the epigraphical multifunction (14) agrees with the standard one, $E_\varphi(\bar{x}) = \{ \mu \in \mathbb{R} \mid \mu \geq \varphi(\bar{x}) \}$, and the subdifferentials (15)–(17) reduce to their well-known prototypes, namely:

- construction (15) with $\varepsilon = 0$ reduces to the Fréchet subdifferential $\hat{\partial}_0 \varphi(\bar{x})$ in (5)—due to the geometric representation (8) of the latter;

- the normal subdifferential (16) reduces to the basic/limiting subdifferential $\partial \varphi(\bar{x})$ by Mordukhovich (2006)

- the singular subdifferential in (17) reduces to $\partial^\infty \varphi(\bar{x})$ in Mordukhovich (2006a).

Among the strongest advantages of the coderivative approach to subdifferentials of set-valued and vector-valued mappings is full coderivative calculus of Mordukhovich (2006a), which induces a variety of calculus rules for the subdifferential constructions defined in (15)–(17). Other major advantages include...
complete coderivative characterizations of fundamental properties in nonlinear analysis related to metric regularity, linear openness, and robust Lipschitzian stability of set-valued mappings; see Mordukhovich (2006a), Rockafellar and Wets (1998). These characterizations generate the corresponding results for mappings with values in partial ordered spaces via the subdifferentials (15)--(17) introduced in this paper.

In infinite-dimensional spaces, the afore-mentioned calculus and characterizations require certain additional “sequential normal compactness” properties of sets and mappings, which are automatic in finite dimensions, while turn out to be a crucial ingredient of variational analysis in infinite dimensions; see the books Mordukhovich (2006a,b) for a comprehensive theory and numerous applications of various properties of this type.

Let us recall some of these properties needed in the paper. Considering generally a set $\Omega \subset X \times Z$ in the product of Banach spaces, we say that it is sequentially normally compact (SNC) at $\bar{v} = (\bar{x}, \bar{z}) \in \Omega$ if for any sequences $\varepsilon_k \downarrow 0$, $v_k \Omega \to \bar{v}$, and $(x^*_k, z^*_k) \in \mathcal{N}_{\varepsilon_k}(v_k; \Omega)$, \(k \in \mathbb{N}, \) one has the implication $(x^*_k, z^*_k) \overset{w^*}{\to} 0 \implies \|x^*_k\| \to 0$ as $k \to \infty$. The product structure of the space in question plays no role in this property (we can put $Z = \{0\}$ without loss of generality) in contrast to its following partial modifications. We say that $\Omega$ is partially sequentially normally compact (PSNC) with respect to $X$ at $\bar{v} \in \Omega$ if for any sequences $(\varepsilon_k, v_k, x^*_k, z^*_k)$ satisfying (18) one has the implication

$$\left[x^*_k \overset{w^*}{\to} 0, \|z^*_k\| \to 0\right] \implies \|x^*_k\| \to 0 \text{ as } k \to \infty.$$

Finally, a set $\Omega$ is strongly PSNC with respect to $X$ at $\bar{v}$ if for any sequences $(\varepsilon_k, v_k, x^*_k, z^*_k)$ satisfying (18) one has $(x^*_k, z^*_k) \overset{w^*}{\to} 0 \implies \|x^*_k\| \to 0$ as $k \to \infty$. We can equivalently put $\varepsilon_k = 0$ in (18) for all the above properties if both spaces $X$ and $Z$ are Asplund and if the set $\Omega$ is locally closed around $\bar{v}$.

Given a set-valued mapping $F: X \rightrightarrows Z$ between Banach spaces, its SNC/PSNC properties at $(\bar{x}, \bar{z}) \in \text{gph } F$ induce by the corresponding properties of its graph. In particular, we say that $F$ is PSNC at $(\bar{x}, \bar{z})$ if its graph is PSNC with respect to $X$ at this point. The reader can find in Mordukhovich (2006a) a number of efficient conditions for the fulfillment of SNC/PSNC properties of sets and mappings, which often relate to their Lipschitzian behavior of some kind. Furthermore, there is a well-developed SNC calculus in Mordukhovich (2006a) ensuring the preservation of SNC and PSNC properties under natural operations performed on sets and mappings.

For mappings $F: X \rightrightarrows Z$ with values in Banach spaces $Z$ partially ordered by convex cones $\Theta \subset Z$ as in (13), the above SNC and PSNC properties induce the corresponding epigraphical counterparts by applying to their epigraphical multifunctions (14). Following this way, we say that such a mapping $F$ is
sequentially normally epicompact (SNEC) or, respectively, partially SNEC at \((\bar{x}, \bar{z}) \in \text{epi} F\) if the epigraphical multifunction \(\mathcal{E}_F\) is SNC (respectively PSNC) at this point.

Next, we formulate a robust Lipschitzian property of set-valued mappings with values in ordered Banach spaces, which ensures simultaneously the partial SNEC property and the triviality of the singular subdifferential (17), which are both important in what follows. We say that a set-valued mapping \(F: X \rightrightarrows Z\) is epi-Lipschitz-like (ELL) around a point \((\bar{x}, \bar{z}) \in \text{epi} F\) if the associated epigraphical multifunction (14) is Lipschitz-like (or enjoys Aubin’s “pseudo-Lipschitzian” property; see Mordukhovich, 2006a; Rockafellar and Wets, 1998 around this point, i.e., there are neighborhoods \(U\) of \(\bar{x}\) and \(V\) of \(\bar{z}\) and a number \(\ell \geq 0\) such that one has the inclusion

\[
\mathcal{E}_F(x) \cap V \subset \mathcal{E}_F(u) + \ell \|x - u\|I_B \quad \text{for all} \quad x, u \in U.
\]

This robust Lipschitzian property of \(\mathcal{E}_F\) is known to be equivalent to both metric regularity and linear openness properties of the inverse multifunction.

**Proposition 1** (singular subdifferential and partial SNEC property of ELL mappings) Let \(F: X \rightrightarrows Z\) be a mapping between Banach spaces, where \(Z\) is ordered by a cone \(\Theta\). Assume that \(F\) is ELL around \((\bar{x}, \bar{z}) \in \text{epi} F\). Then, \(F\) is partially SNEC at \((\bar{x}, \bar{z})\), and one has the singular subdifferential condition

\[
\partial^\infty F(\bar{x}, \bar{z}) = \{0\}.
\]

**Proof.** The partial SNEC property of \(F\) follows from Mordukhovich (2006a, Proposition 1.68) due to the above definitions of this and ELL properties, while the subdifferential condition (19) as a consequence of definition (17) and Mordukhovich (2006a, Theorem 1.44).

Finally in this section, we formulate the (approximate) extremal principle from Mordukhovich (2006a, Chapter 2), which is the main driving force for the development of the afore-mentioned calculus results and characterizations (including the SNC calculus in infinite dimensions) and plays a crucial role in this paper. Given two sets \(\Omega_1, \Omega_2 \subset X\) locally closed around \(\bar{x} \in \Omega_1 \cap \Omega_2\), we say that \(\bar{x}\) is a local extremal point of the set system \(\{\Omega_1, \Omega_2\}\) if there is a neighborhood \(U\) of \(\bar{x}\) such that for any \(\varepsilon > 0\) there is \(a \in \varepsilon I_B\) with

\[
(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset.
\]

**The Extremal Principle.** Let \(\bar{x}\) be a local extremal point of the set system \(\{\Omega_1, \Omega_2\}\) in the Asplund space \(X\), and let both \(\Omega_1\) and \(\Omega_2\) be locally closed around \(\bar{x}\). Then for any \(\varepsilon > 0\) there are \(x_i \in \Omega_i \cap (\bar{x} + \varepsilon I_B)\) and \(x^*_i \in \tilde{N}(x_i; \Omega_i) + \varepsilon I_B^*\), \(i = 1, 2\), such that

\[
\|x^*_1\| + \|x^*_2\| = 1, \quad x^*_1 + x^*_2 = 0.
\]
3. Variational principles for set-valued mappings

In this section we derive two major variational principles that are extensions of
the variational principles discussed in Section 1 from scalar functions to vector-
valued and set-valued mappings. Let us start with an appropriate extension of
the Ekeland variational principle, which is of undoubted interest for its own sake
and is used in what follows for deriving a required extension of the subdifferen-
tial variational principle to set-valued mappings in terms of the subdifferential
constructions introduced in Section 2.

It is well understood that the conventional proof of the classical Ekeland vari-
ational principle for extended-real-valued functions (see Ekeland, 1974, 1979 and
also Mordukhovich, 2006a, Theorem 2.26) cannot be directly extended to the
vector and set-valued mappings with merely partially ordered (while not totally
ordered) range spaces. Several approaches to vector/set-valued extensions of
this fundamental result and its proof are suggested in the literature (based on
certain vector metrics, scalarization techniques, etc.—compare Bao and Khanh,
2003; Chen, Huang and Yang, 2005; Göpfert et al., 2003; Ha, 2005; Hamel,
2005; Hamel and Löhne, 2006; Khanh, 1989 for more details, discussions, and
references), but unfortunately they do not allow us to arrive at all the conclu-
sions needed for our purposes; see below. Our proof is based on a new iterative
procedure, which does not involve any scalarization technique and deals directly
with the set/vector-valued setting under consideration.

To formulate a set-valued extension of the Ekeland variational principle, we
first recall some relevant notions from set-valued optimization mainly following

Let \((X, d)\) be a complete metric space, and let \(Z\) be a partially ordered linear
topological space, where the partial order \(\leq\) is generated by a closed and
convex cone \(\Theta\) via (13). In what follows we always assume that the ordering
cone \(\Theta\) is pointed, i.e., \(\Theta \cap (-\Theta) = \{0\}\).

Given a set \(\Lambda \subset Z\) and a point \(\bar{z} \in \Lambda\), we say that \(\bar{z}\) is a minimal point of \(\Lambda\) if
\[\Lambda \cap (\bar{z} - \Theta) = \{\bar{z}\}\].

The collection of minimum points to \(\Lambda\) can be equivalently described by
\[\text{Min } \Lambda := \{\bar{z} \in \Lambda | \bar{z} - z \notin \Theta \text{ whenever } z \in \Lambda, \ z \neq \bar{z}\}\].

If \(\text{int } \Theta \neq \emptyset\), we similarly consider weak minimal points \(\bar{z}\) of \(\Lambda\) defined by
\[\Lambda \cap (\bar{z} - \text{int } \Theta) = \emptyset\].

A set-valued mapping \(F : X \Rightarrow Z\) is epiclosed if its epigraph with respect to
the ordering cone \(\Theta\) is closed in \(X \times Z\). This mapping is level-closed if for all
\(z \in Z\) its \(z\)-level set
\[\mathcal{L}(z) := \{x \in X | \exists v \in F(x) \text{ with } v \leq z\}\].
is closed in $X$. It is clear that every epiclosed mapping is level-closed but not vice versa. We say that the set-valued mapping $F$ is \textit{quasibounded from below} if there exists a bounded and closed subset $M \subset Z$ such that
\[ F(X) \subset M + \Theta, \]
and that the mapping $F$ is \textit{bounded from below} if the set $M$ above can be chosen as a singleton. Correspondingly, a set $\Omega \subset Z$ is quasibounded (bounded) from below if the constant mapping $F(x) \equiv \Omega$ has this property.

Now, having a mapping $F: X \to Z$ from a complete metric space $(X, d)$ to a partially ordered linear topological space $Z$ with the ordering cone $\Theta$, we consider the \textit{set-valued optimization problem}
\[
\text{minimize } F(x) \text{ subject to } x \in X \tag{20}
\]
with no explicit constraints on $x$, although they are hidden by $x \in \text{dom } F := \{ x \in X \mid F(x) \neq \emptyset \}$.

In this paper we study the following notions of exact and approximate minimizers to set-valued and vector-valued mappings.

\textbf{Definition 2 (minimizers and approximate minimizers in set-valued optimization)} \textit{Given a mapping $F: X \to Z$ taking values in a partially ordered space with the ordering cone $\Theta$, we consider the set-valued minimization problem (20) and say that:}

\begin{enumerate}[(i)]
  \item $(\bar{x}, \bar{z}) \in \text{gph } F$ is a \textit{minimizer} to (20)—or just to the mapping $F$—if $\bar{z} \in F(\bar{x})$ is a minimal point of the image set $F(X) := \bigcup_{x \in X} F(x)$, i.e.,
  \[ (\bar{z} - \Theta) \cap F(X) = \{ \bar{z} \}. \tag{21} \]
  \item $(\bar{x}, \bar{z})$ is a \textit{weak minimizer} to (20) if $\bar{z} \in F(\bar{x})$ is a weak minimum point of the set $F(X)$, i.e., (21) holds with the replacement of $\Theta$ by $\text{int } \Theta \neq \emptyset$ and $\{ \bar{z} \}$ by $\emptyset$.
  \item Given $\varepsilon > 0$ and $\xi \in \Theta \setminus \{0\}$, we say that $(\bar{x}, \bar{z}) \in \text{gph } F$ is an \textit{approximate $\varepsilon\xi$-minimizer} to (20) if
  \[ z + \varepsilon \xi \not\leq \bar{z} \text{ for all } z \in F(x) \text{ with } x \neq \bar{x}. \]
  \item $(\bar{x}, \bar{z})$ is an \textit{approximate $\varepsilon\xi$-minimizer} to (20) if there is a positive number $\tilde{\varepsilon} < \varepsilon$ such that $(\bar{x}, \tilde{z})$ is an approximate $\tilde{\varepsilon}\xi$-minimizer to this problem.
\end{enumerate}

Now we are ready to formulate and prove our set-valued extension of the Ekeland variational principle. Following Luc (1989), we say that $F$ has the \textit{domination property} if

\[ \text{for every } x \in X \text{ and } z \in F(x) \text{ there is } \tilde{z} \in \text{Min } F(x) \text{ with } \tilde{z} \leq z, \tag{22} \]

where the minimum set $\text{Min } F(x)$ is closed. In Luc (1989) and the reference therein, the reader can find some efficient conditions ensuring this property.
THEOREM 1 (Ekeland variational principle for set-valued mappings) Let \((X,d)\) be a complete metric space, and let \(Z\) be a partially ordered linear topological space with order \((13)\) generated by a convex, closed, and pointed cone \(\Theta \neq \{0\}\). Consider a set-valued mapping \(F: X \to Z\) and assume that \(F\) is quasibounded from below, level-closed, and has the domination property \((22)\). Then for any \(\varepsilon > 0\), \(\lambda > 0\), \(\xi \in \Theta\) with \(\|\xi\| = 1\), and \((x_0, z_0) \in gph F\) there is a point \((\bar{x}, \bar{z}) \in gph F\) satisfying the relationships

\[
\bar{z} - z_0 + \frac{\varepsilon}{\lambda} d(\bar{x}, x_0) \xi \leq 0, \quad \bar{z} \in \Min F(\bar{x}),
\]

\[
\bar{z} - z + \frac{\varepsilon}{\lambda} d(\bar{x}, x) \xi \leq 0 \quad \text{for all } (x, z) \in gph F \text{ with } (x, z) \neq (\bar{x}, \bar{z}).
\]

If \((x_0, z_0)\) is an approximate \(\varepsilon\)-\(\xi\)-minimizer to \(F\), then \(\bar{x}\) can be chosen such that in addition to \((23)\) and \((24)\) we have

\[
d(\bar{x}, x_0) \leq \lambda.
\]

**Proof.** Note first that it is sufficient to prove the theorem in the case of \(\varepsilon = \lambda = 1\). Indeed, the general case can be easily reduced to this special case by applying the latter to the mapping \(\hat{F}(x) := \varepsilon^{-1} F(x)\) on the metric space \((X,d)\) with \(d(x, y) := \lambda^{-1} d(x, y)\).

Having this in mind, introduce a set-valued mapping \(T: X \times Z \to X\) by

\[
T(x, z) := \{ y \in X | \exists v \in F(y) \text{ with } v - z + d(x, y) \xi \leq 0 \}
\]

and observe that \(T\) has the following properties:

- The sets \(T(x, z)\) are nonempty for all \(z \in F(x)\), since \(x \in T(x, z)\).
- The sets \(T(x, z)\) are closed for all \(z \in F(x)\), since the mapping \(F\) is level-closed.
- The sets \(T(x, z)\) are uniformly bounded for all \(z \in F(x)\), since the mapping \(F\) is quasibounded from below. Indeed, one has

\[
T(x, z) \subset \{ y \in X | d(x, y) \xi \in z - M - \Theta \},
\]

where the bounded set \(M\) is taken from the above definition of quasiboundedness of \(F\) from below.

- One has the inclusion \(T(y, v) \subset T(x, z)\) for all \(y \in T(x, z)\) and \(v \in F(y)\) with

\[
v - z + d(x, y) \xi \leq 0.
\]

Indeed, pick \(u \in T(y, v)\) and by construction of \(T\) in \((26)\) find \(w \in F(u)\) satisfying the inequality \(w - v + d(y, u) \xi \leq 0\). Summing the last two inequalities and taking into account that \(d(x, y) + d(y, u) \geq d(x, u), \xi \in \Theta,\) and \(\Theta + \Theta \subset \Theta,\) we have

\[
w - z + d(x, u) \xi = (w - v + d(y, u) \xi) + (v - z + d(x, y) \xi) + (d(x, u) - d(y, u) - d(x, y)) \xi \in -\Theta - \Theta - \Theta \subset -\Theta,
\]

which implies that \(u \in T(x, z)\).
Let us inductively construct a sequence of pairs \( \{(x_k, z_k)\} \subset \text{gph} \ F \) by the following iterative procedure: starting with \((x_0, z_0)\) given in theorem and having the \(k\)-iteration \((x_k, z_k)\), we select the next one \((x_{k+1}, z_{k+1})\) by

\[
\begin{cases}
  x_{k+1} \in T(x_k, z_k), \\
  d(x_k, x_{k+1}) \geq \sup_{x \in T(x_k, z_k)} d(x_k, x) - \frac{1}{k + 1}, \\
  z_{k+1} \in F(x_{k+1}), \quad z_{k+1} - z_k + d(x_k, x_{k+1})\xi \leq 0.
\end{cases}
\]  

(27)

It is clear from the construction and properties of \(T(x, z)\) in (26) that the iterative procedure (27) is well defined. Summing up the last inequality in (27) from \(k = 0\) to \(m\), we get

\[
\left( \sum_{k=0}^{m} d(x_k, x_{k+1}) \right) \xi \in z_0 - z_{m+1} - \Theta \subset z_0 - M - \Theta
\]

and, by passing to the limit as \(m \to \infty\) and using the quasiboundedness of the mapping \(F\) from below and the pointedness of the ordering cone \(\Theta\) with \(0 \neq \xi \notin -\Theta\), we arrive at the conclusions

\[
\left( \sum_{k=0}^{\infty} d(x_k, x_{k+1}) \right) \xi \in z_0 - M - \Theta \quad \text{and} \quad \sum_{k=0}^{\infty} d(x_k, x_{k+1}) < \infty.
\]

Taking then into account that \(\text{diam} \ T(x_{k+1}, z_{k+1}) \leq \text{diam} \ T(x_k, z_k)\) and the choice of \(x_{k+1}\), we have the estimate

\[
\text{diam} \ T(x_k, z_k) \leq 2 \sup_{x \in T(x_k, z_k)} d(x_k, x) \leq 2\left(d(x_k, x_{k+1}) + \frac{1}{k + 1}\right),
\]

and hence \(\text{diam} \ T(x_k, z_k) \downarrow 0\) as \(k \to \infty\). Due to the completeness of \(X\) we conclude that the sets \(T(x_k, z_k)\) shrink to a singleton:

\[
\bigcap_{k=0}^{\infty} T(x_k, z_k) = \{\bar{x}\} \quad \text{with some} \; \bar{x} \in X.
\]  

(28)

Let us next justify the existence of \(\bar{z} \in F(\bar{x})\) such that \((\bar{x}, \bar{z})\) satisfies relationships (23) and (24). For each \(z_k \in Z\) from (27) define the set

\[
R(x_k, z_k) := \{z \in \text{Min} F(\bar{x}) \mid z - z_k + d(x_k, \bar{x})\xi \leq 0\}, \quad k = 0, 1, \ldots
\]

(29)

Then we have the following properties:

- The set \(R(x_k, z_k)\) is nonempty and closed for any \(k = 0, 1, \ldots\) by the assumptions made in the theorem. Indeed, it is easily implied by the last line in (27) that whenever \(m \geq 1\) one has \(x_{k+m} \in L(z_k - d(x_k, \bar{x})\xi)\) for the level set of \(F\), which is assumed to be closed. Hence \(\bar{x} \in L(z_k - d(x_k, \bar{x})\xi)\), i.e., there is \(\bar{z} \in F(\bar{x})\) satisfying \(\bar{z} - z_k + d(x_k, \bar{x})\xi \leq 0\).
Furthermore, by the domination condition described in (22) there is element \( \hat{z} \in \text{Min} F(\bar{x}) \) with \( \bar{z} \leq \hat{z} \). Taking into account the previous inequality, we get \( \hat{z} \in R(x_k, z_k) \), i.e., \( R(x_k, z_k) \neq \emptyset \). The closedness of the set \( R(x_k, z_k) \) follows directly from that of \( \text{Min} F(\bar{x}) \) by construction (29).

- The set sequence \( \{R(x_k, z_k)\} \) is nonincreasing, i.e., \( R(x_{k+1}, z_{k+1}) \subset R(x_k, z_k) \) for all \( k = 0, 1, \ldots \). To justify this, pick any \( z \in R(x_{k+1}, z_{k+1}) \) and observe that

\[
\hat{z} - z_{k+1} + d(x_{k+1}, \bar{x}) \xi \leq 0.
\]

Adding the latter inequality to the one in (27), we have \( \hat{z} - z_k + d(x_k, \bar{x}) \xi \leq 0 \), i.e., \( z \in R(x_k, z_k) \) for all \( k = 0, 1, \ldots \).

It follows from the above properties that

\[
\emptyset \neq \bigcap_{k=0}^{\infty} R(x_k, z_k) \subset \text{Min} F(\bar{x}).
\]

Take an arbitrary vector \( \bar{z} \) from the above intersection and by using domination property (22) show that the pair \((\bar{x}, \bar{z}) \in \text{gph} F\) satisfies relationships (23) and (24). Indeed, the one in (23) immediately follows from \( \bar{z} \in R(x_0, z_0) \) and the construction of \( R(\cdot, \cdot) \) in (29). To justify (24), suppose that it does not hold and then find a point

\[
(x, z) \in \text{gph} F \text{ with } (x, z) \neq (\bar{x}, \bar{z}) \text{ and } z - \bar{z} + d(x, \bar{x}) \xi \leq 0. \tag{30}
\]

If \( x = \bar{x} \) in (30), then we obviously have \( z \leq \bar{z} \), which contradicts the minimality of \( \bar{z} \) on the set \( F(\bar{x}) \). If \( x \neq \bar{x} \), then it follows from (30) and the construction of \( \bar{z} \) that

\[
z - \bar{z} + d(\bar{x}, x) \xi \leq 0 \quad \text{and} \quad \bar{z} - z_k + d(\bar{x}, x_k) \xi \leq 0, \quad k = 0, 1, \ldots.
\]

Summing up the last two inequalities and combining this with the triangle one, we get

\[
z - z_k + d(x, x_k) \xi \leq 0, \quad \text{i.e., } x \in T(x_k, z_k) \text{ for all } k = 0, 1, \ldots.
\]

This means that \( x \) from (30) belongs to the set intersection in (28). Thus \( x = \bar{x} \) by (28), which justifies (24).

To complete the proof of the theorem, it remains to estimate the distance \( d(\bar{x}, x_0) \) when \((x_0, z_0)\) is an approximate \( \varepsilon \xi \)-minimizer to \( F \). Arguing by contradiction, suppose that (25) does not hold, i.e., \( d(\bar{x}, x_0) > \lambda \). Since \( \bar{x} \in T(x_0, z_0) \), we have

\[
\bar{z} - z_0 + \varepsilon \xi \leq \bar{z} - z_0 + \frac{\varepsilon}{\lambda} d(\bar{x}, x_0) \xi \leq 0,
\]

which contradicts the approximate minimum assumption on \((x_0, z_0)\). Thus (25) holds, and the proof of the theorem is finished. \( \blacksquare \)
Note that, by the order definition (13), conclusion (23) of Theorem 1 immediately implies that \( \bar{z} \leq z_0 \). When \( F = f : X \to Z \) is single/vector-valued, we have the following corollary (and simplification) of Theorem 1, which agrees with the classical Ekeland variational principle for scalar functions.

**Corollary 1 (Ekeland variational principle for vector-valued mappings)** Let \((X, d), Z, \) and \( \Theta \) be as in Theorem 1, and let \( f : X \to Z \) be a single-valued mapping, which is level-closed and quasibounded from below. Take any \( \varepsilon > 0, \lambda > 0, \xi \in \Theta \setminus \{0\} \), and \( x_0 \in X \) that is assumed to be an approximate \( \varepsilon \xi \)-minimizers to \( f \), i.e.,

\[
f(x) + \varepsilon \xi \not\leq f(x_0) \quad \text{whenever} \quad x \in X.
\]

Then there is \( \bar{x} \in X \) such that \( d(\bar{x}, x_0) \leq \lambda, f(\bar{x}) \leq f(x_0) \), and

\[
f(x) - f(\bar{x}) + \frac{\varepsilon}{\lambda} d(\bar{x}, x) \xi \not\leq 0 \quad \text{for all} \quad x \in X \setminus \{\bar{x}\}.
\]

**Proof.** It follows directly from Theorem 1 by observing that \( \text{Min} \ f(x) \neq \emptyset \) whenever \( x \in X \) for single-valued mappings. In this case the part in the proof of Theorem 1 related to considering the sets \( R(x, z) \) in (28) is not needed.

**Remark 1** (comparison with other extensions of the Ekeland principle) Note that the proofs of Theorem 1 and Corollary 1, based on the iteration technique (27) involving the mapping \( T(x, z) \) in (26), do not use any scalarization and/or vector metric as in Chen, Huang and Yang (2005), Göpfert et al. (2003), Ha (2003, 2005, 2006), Hamel (2005), Hamel and Löhne (2006), Khánh (1989), and do not impose any assumptions on nonempty interior, upper semicontinuity/demicontinuity, compactness, boundedness from below (instead of quasi-boundedness from below), etc. as in many previous results.

It is easy to check that the principal relationship (24) can be equivalently rewritten as

\[
z - \bar{z} + \frac{\varepsilon}{\lambda} d(x, \bar{x}) \xi \not\leq 0 \quad \text{for all} \quad z \in F(\bar{x}), z \neq \bar{z}.
\]

In comparison we observe that the corresponding principal condition of Hamel and Löhne (2006) can be written in our setting as

\[
z \not\in \bar{z} - \frac{\varepsilon}{\lambda} d(\bar{x}, x) \xi - \Theta \quad \text{for all} \quad (x, z) \in \text{gph} \ F \quad \text{with} \quad x \neq \bar{x}, \quad (31)
\]

while the one in Ha (2005) is equivalent to

\[
F(\bar{x}) \not\subset F(x) + \frac{\varepsilon}{\lambda} D(\bar{x}, x) \xi + \Theta \quad \text{for all} \quad x \in X \quad \text{with} \quad x \neq \bar{x}. \quad (32)
\]

We can easily see that (24) \( \implies \) (31) \( \implies \) (32). Furthermore, (24) is strictly better than both (31) and (32). A simple example is provided by \( F : \mathbb{R} \to \mathbb{R} \) given as

\[
F(x) := \begin{cases} [-1, 1] & \text{for} \ x = 0, \\ 0 & \text{otherwise} \end{cases}
\]
with $\Theta = \mathbb{R}_+$, we see that $\bar{x} = 0$ satisfies (31) and $(\bar{x}, \hat{z}) = (0, 0)$ satisfies (32) while not (24). On the other hand, relationship (31) together with the domination property (22) imply (24). Indeed, by (22) there is $\hat{z} \in \text{Min } F(\bar{x})$ such that $\hat{z} \leq \bar{z}$, and hence
\[
\hat{z} - \frac{\varepsilon}{\lambda} d(\bar{x}, x) \xi - \Theta \subset \bar{z} - \frac{\varepsilon}{\lambda} d(\bar{x}, x) \xi - \Theta.
\]
The latter inclusion together with (31) give
\[
z \notin \hat{z} - \frac{\varepsilon}{\lambda} d(\bar{x}, x) \xi - \Theta \quad \text{for all } z \in F(x), \ x \neq \bar{x},
\]
which implies (24) for $(\bar{x}, \hat{z})$ due to $\hat{z} \in \text{Min } F(\bar{x})$.

We finally emphasize that the new condition (24) plays a crucial role in the proof of the following Theorem 2: it allows us to organize an extremal system of sets (see the proof), which does not seem to be possible by using conditions (31) and (32).

Next we establish a new set-valued extension of the subdifferential variational principle from Mordukhovich (2006a), Mordukhovich and Wang (2002) by using the Fréchet subdifferential of set-valued/vector-valued mappings introduced in Definition 1. Previous versions of this result, which either follow from our theorem or are different from it in both assumptions and conclusions, can be found in Ha (2003, 2005, 2006).

**Theorem 2** (subdifferential variational principle for set-valued mappings) Let $F : X \rightrightarrows Z$ be a set-valued mapping between Asplund spaces that is epiclosed, quasibounded from below and satisfies the domination condition (22), where the ordering cone $\Theta$ of $Z$ satisfies the assumptions of Theorem 1. Take any $\varepsilon > 0$, $\lambda > 0$, $\xi \in \Theta$ with $\|\xi\| = 1$ and consider a strict approximate $\varepsilon\xi$-minimizer $(x_0, z_0) \in \text{gph } F$ to the mapping $F$. Then there is $(\bar{x}, \bar{z}) \in \text{gph } F$ such that $\|\bar{x} - x_0\| \leq \lambda$ and the subdifferential condition
\[
\hat{\partial} F(\bar{x}, \bar{z}) \cap \frac{\varepsilon}{\lambda} B^* \neq \emptyset
\]
is satisfied. If, furthermore, $\xi \in \text{int } \Theta$, then the pair $(\bar{x}, \bar{z})$ above can be selected as an approximate $\varepsilon\xi$-minimizer to $F$.

**Proof.** Since $(x_0, z_0)$ is a strict approximate $\varepsilon\xi$-minimizer to $F$, there is a positive number $\bar{\varepsilon} < \varepsilon$ such that $(x_0, z_0)$ is an approximate $\bar{\varepsilon}\xi$-minimizer to $F$. Define the number
\[
\bar{\lambda} := \frac{\varepsilon + \bar{\varepsilon}}{2} \lambda \quad \text{with } 0 < \bar{\lambda} < \lambda
\]
and apply to the mapping $F$ and its approximate $\bar{\varepsilon}\xi$-minimizer $(x_0, z_0)$ the generalized Ekeland variational principle from Theorem 1 with the parameters $(\bar{\varepsilon}, \bar{\lambda})$. Then we find by (23)–(25) a point $(\bar{u}, \bar{v}) \in \text{gph } F$ satisfying the conditions
\[
\bar{v} - z_0 + \frac{\bar{\varepsilon}}{\lambda} \|x_0 - \bar{u}\| \xi \in -\Theta, \quad \|x_0 - \bar{u}\| \leq \bar{\lambda},
\]
z - \bar{v} + \frac{\bar{\varepsilon}}{\lambda} \|x - \bar{u}\| \xi \notin -\Theta \text{ for all } (x, z) \in \operatorname{gph} F \text{ with } (x, z) \neq (\bar{u}, \bar{v}). \quad (36)

Define further a set-valued mapping \( G: X \rightrightarrows Z \) by

\[
G(x) := \bar{v} - \frac{\bar{\varepsilon}}{\lambda} \|x - \bar{u}\| \xi - \Theta
\]

and consider the following two closed subsets of the product space \( X \times Z \) (which is well known to be Asplund; see, e.g., Phelps, 1993):

\[
\Omega_1 := \operatorname{epi} F \quad \text{and} \quad \Omega_2 := \operatorname{gph} G.
\]

Let us check that \((\bar{u}, \bar{v})\) is an extremal point of the set system \( \{\Omega_1, \Omega_2\} \) from \( (38) \) in the sense of Mordukhovich (2006a, Definition 2.1); see the formulation at the end of Section 2. Indeed, we obviously have \((\bar{u}, \bar{v}) \in \Omega_1 \cap \Omega_2\), and thus the extremality of this system follows from the fact that

\[
\Omega_1 \cap \left( \Omega_2 + (0, -k^{-1}\zeta) \right) = \emptyset \quad \text{for all } k \in \mathbb{N}, \quad (39)
\]

where \( \zeta \neq 0 \) is an arbitrary fixed element of the cone \( \Theta \). Suppose that \( (39) \) does not hold for some fixed \( k \in \mathbb{N} \). By the constructions of \( \Omega_1 \) and \( \Omega_2 \) in \( (38) \) and the fact that

\[
\operatorname{epi} F = \{(x, w) \in X \times Z | \exists z \in Z, \exists \theta \in \Theta \text{ with } w = z + \theta, (x, z) \in \operatorname{gph} F\}
\]

our assumption means that there are \((x, z + \theta) \neq (\bar{u}, \bar{v})\) and \( \theta \in \Theta \) such that

\[
(x, z + \theta) \in \operatorname{epi} F \quad \text{and} \quad (x, z + \theta + k^{-1}\zeta) \in \operatorname{gph} G.
\]

By the structure of \( G \) in \( (37) \) and the convexity of the cone \( \Theta \) we have

\[
z + \theta + \frac{\zeta}{k} \in \bar{v} - \frac{\bar{\varepsilon}}{\lambda} \|x - \bar{u}\| \xi - \Theta
\]

and hence

\[
z - \bar{v} + \frac{\bar{\varepsilon}}{\lambda} \|x - \bar{u}\| \xi \in -\theta - \frac{\zeta}{k} - \Theta \subset -\Theta
\]

for the point \((x, z) \in \operatorname{gph} F\) under consideration, which implies by \( (36) \) that \((x, z) = (\bar{u}, \bar{v})\). Since \( \bar{v} + \bar{\varepsilon} = z + \theta = F(x) = F(\bar{u}) \), we get from \( (36) \) that \( \bar{v} + \theta - \bar{v} = \theta \leq 0 \), and so \( \theta = 0 \). This clearly contradicts the above relationship \((x, z + \theta) \neq (\bar{u}, \bar{v})\) and justifies therefore the extremality of the system \( (38) \) at \((\bar{u}, \bar{v})\).

Thus, we can apply the extremal principle from Mordukhovich (2006a, Theorem 2.20) (this result is presented at the end of Section 2) to the extremal system \( \{\Omega_1, \Omega_2, (\bar{u}, \bar{v})\} \) of the closed sets \( (38) \) at \((\bar{u}, \bar{v})\) in the Asplund space \( X \times Z \) with the norm \( \| (x, z) \| := \| x \| + \| z \| \) for \((x, z) \in X \times Z \). Observe that the corresponding dual norm on \( X^* \times Z^* \) is

\[
\| (x^*, z^*) \| = \max \{ \| x^* \|, \| z^* \| \} \quad \text{for} \quad (x^*, z^*) \in X^* \times Z^*.
\]
Employing the extremal principle, for any $\nu > 0$ we find $(x_i, z_i, x_i^*, z_i^*) \in X \times Z \times X^* \times Z^*$ with $i = 1, 2$ satisfying the relationships

$$\begin{cases} (x_i, z_i) \in \Omega_1 \times \Omega_2, & \|x_i - \bar{u}\| + \|z_i - \bar{v}\| \leq \nu, \quad i = 1, 2, \\
x_i^*, z_i^* \in \hat{N}(x_i, z_i; \Omega_i), & i = 1, 2, \\
\frac{1}{2} - \nu \leq \max \left\{ \|x_i^*\|, \|z_i^*\| \right\} \leq \frac{1}{2} + \nu, \quad i = 1, 2, \\
\max \left\{ \|x_i^* + x_2^*\|, \|z_i^* + z_2^*\| \right\} \leq \nu. \end{cases} \tag{40}$$

Observe that $(x_2^*, z_2^*) \neq 0$ whenever $\nu > 0$ is sufficiently small in (40). It also follows from the second line in (40), the graphical structure of the set $\Omega_2$ in (38), and the coderivative construction (10) as $\varepsilon = 0$ that

$$x_2^* \in \hat{D}^* G(x_2, z_2)(-z_2^*). \tag{41}$$

To proceed further with (40) and (41), let us check that the set-valued mapping $G$ is Lipschitz continuous on $X$ with the (global) Lipschitz constant $\ell := \bar{\ell}/\hat{\lambda}$, i.e.,

$$G(x) \subset G(y) + \ell \|x - y\| \mathcal{B} \quad \text{whenever} \quad x, y \in X. \tag{42}$$

To justify (42), take any $z \in G(x)$ and find by (37) and the definition of $\ell$ such $\xi \in \Theta$ that $z = \bar{v} - \ell \|x - \bar{u}\| \xi - \zeta$. Then we have the following relationships:

$$\begin{align*}
z &= \bar{v} - \ell \|x - \bar{u}\| \xi - \zeta \\
    &= \bar{v} - \ell \|y - \bar{u}\| \xi + \ell \|x - y\| \xi + \ell \|x - \bar{u}\| - \|y - \bar{u}\| - \|x - y\| \xi - \zeta \\
    &\subset \bar{v} - \ell \|y - \bar{u}\| \xi + \ell \|x - y\| \xi - \Theta - \zeta \\
    &\subset \bar{v} - \ell \|y - \bar{u}\| \xi - \Theta + \ell \|x - y\| \xi = G(y) + \ell \|x - y\| \xi,
\end{align*}$$

where the first inclusion holds due to

$$\|x - \bar{u}\| - \|y - \bar{u}\| - \|x - y\| \leq 0 \quad \text{and} \quad \xi \in \Theta$$

and the second one holds due to the convexity of $\Theta$. Since $\|\xi\| = 1$, we arrive at (42).

Employing now the coderivative estimate for Lipschitzian mappings from Mordukhovich (2006a, Theorem 1.43), we get from (41) that

$$\|x_2^*\| \leq \ell \|z_2^*\| \quad \text{and hence} \quad \|z_2^*\| \neq 0, \quad \frac{\|x_2^*\|}{\|z_2^*\|} \leq \ell \tag{43}$$

by the third line in (40) for $i = 2$. Furthermore, it gives

$$\|z_2^*\| \geq \min \left\{ \ell \left( \frac{1}{2} - \nu \right), \left( \frac{1}{2} - \nu \right) \right\}. \tag{44}$$

This inequality, together with the last line of (40), ensure that $z_1^* \neq 0$ whenever $\nu$ is sufficiently small. Then, by the structure of $\Omega_1$ and the second line of (40) for $i = 1$, we have $(x_1^*, z_1^*) \in \hat{N}((x_1, z_1)); \text{epi} F$), which implies—by the
construction of the Fréchet normal cone in (7) and the structure of epi $F$—that there is $	ilde{z}_1 \in F(x_1)$ and $\vartheta \in \Theta$ with

$$\tilde{z}_1 = z_1 - \vartheta, \quad (x_1^*, z_1^*) \in \tilde{N}((x_1, \tilde{z}_1); epi F), \quad \text{and} \quad z_1^* \in \tilde{N}(0; \Theta).$$

Taking (10) and (14) into account, we thus have

$$\frac{x_1^*}{\|z_1^*\|} \in \tilde{D}^* F(x_1, \tilde{z}_1) \left(\frac{-z_1^*}{\|z_1^*\|}\right) \quad \text{with} \quad (x_1, \tilde{z}_1) \in gph F.$$

(45)

It follows from (44) that $\nu/\|z_1^*\| \to 0$ as $\nu \downarrow 0$ and that, by the above estimates,

$$\frac{\|x_1^*\|}{\|z_1^*\|} < \frac{\|x_1^*\| + \nu}{\|z_1^*\|} = \left(\frac{\|x_1^*\| + \nu}{\|z_1^*\|}\right) \left(1 - \frac{\nu}{\|z_1^*\|}\right) < \frac{\varepsilon}{\lambda}$$

for all $\nu > 0$ sufficiently small. Observe also that

$$\|x_1 - x_0\| \leq \|\bar{u} - x_0\| + \|x_1 - \bar{u}\| \leq \tilde{\lambda} + \nu < \lambda$$

for all small $\nu > 0$ by the second inequality in (35), the first line in (40) for $i = 1$, and the choice of $\tilde{\lambda}$ in (34). Denoting $(\bar{x}, \bar{z}) := (x_1, \tilde{z}_1)$ and taking into account the subdifferential construction (15), we get from (45) and the subsequent estimates that the desired relationship (33) is satisfied with $\|\bar{x} - x_0\| < \lambda$.

To complete the proof of the theorem, it remains to justify that $(\bar{x}, \bar{z}) = (x_1, \tilde{z}_1)$ is a $\varepsilon \xi$-minimizer to $F$ provided that $\xi \in \text{int } \Theta$. In this case $(\varepsilon - \tilde{\varepsilon})\xi \in \text{int } \Theta$, and for all $\nu > 0$ sufficiently small we obviously have

$$\nu B \subset (\varepsilon - \tilde{\varepsilon})\xi - \Theta.$$

(46)

It follows from the first line of (40) that $\|z_1 - \bar{v}\| \leq \nu$. By (46) we find $\zeta \in \Theta$ such that $z_1 - \bar{v} = (\varepsilon - \tilde{\varepsilon})\xi - \zeta$. If $(x_1, \tilde{z}_1)$ is not an approximate $\varepsilon \xi$-minimizer to $F$, then there is $(x, z) \in gph F$ satisfying

$$z + \varepsilon \xi \in \tilde{z}_1 - \Theta = z_1 - \vartheta - \Theta = \bar{v} + (\varepsilon - \tilde{\varepsilon})\xi - \zeta - \vartheta - \Theta.$$

Since $\bar{v} \in z_0 - \Theta$ by (35), we get in this case that

$$z + \tilde{\varepsilon} \xi \in z_0 - \Theta,$$

which contradicts the strict approximate $\varepsilon \xi$-minimality of the initial pair $(x_0, z_0)$ to $F$ and thus ends the proof of the theorem. □
4. Existence of optimal solutions to multiobjective problems

In this section we study the existence of optimal solutions to the constrained multiobjective (set-valued and vector-valued) optimization problem:

$$\minimize \ F(x) \ \text{subject to} \ x \in \Omega,$$

where $F: X \rightarrow Z$ is a mapping from a complete metric space $(X, d)$ to a partially ordered linear topological space $Z$ with the ordering cone $\Theta \subset Z$ assumed to be closed, convex, and pointed. Our goal in this section is to establish efficient conditions for the existence of weak minimizers to (47), and thus we impose the interiority requirement on the ordering cone: $\text{int} \ \Theta \neq \emptyset$. The afore-mentioned assumptions are standing throughout this section.

In what follows we present three results for the existence of weak minimizers to (47). The first two results unified in one theorem employ our basic construction in the proof of Theorem 1—an extension of the Ekeland variational principle to set-valued mappings. We start with the compactness requirement on the constraint set $\Omega$ and then replace it by a certain coercivity condition imposed on the cost mapping. The third existence result is based on the application of the subdifferential variational principle from Theorem 2 combined with an appropriate subdifferential extension of the Palais-Smale condition and generalized differential calculus rules developed in Mordukhovich (2006a).

**Theorem 3** (existence of weak minimizers under either compactness of constraint sets or coercivity of cost mappings) Consider the constrained multiobjective optimization problem (47) under the standing assumptions made in this section. Suppose also that $F$ is quasibounded from below and level-closed. Then, problem (47) admits a weak minimizer in each of the following cases:

(i) Let the constraint set $\Omega$ be compact, and let the cost mapping $F$ satisfy the limiting monotonicity condition as $k \rightarrow \infty$:

$$[x_k \rightarrow \bar{x}, \ z_k \in F(x_k) \ \text{with} \ z_{k+1} \leq z_k] \implies \exists \ \bar{z} \in \text{Min} \ F(\bar{x}) \ \text{with} \ \bar{z} \leq z_k] \quad (48)$$

for all $k \in \mathbb{N}$; the latter is implied by the domination condition (22) of Theorem 1 provided that $F$ is level-closed.

(ii) Let the cost mapping $F$ satisfy (48) and the coercivity condition: there is a compact set $\Xi \subset X$ such that

$$[x \in X \setminus \Xi, \ z \in F(x)] \implies [\exists (y, v) \in \text{gph} \ F \ \text{with} \ y \in \Xi \ \text{and} \ v \leq z]. \quad (49)$$

**Proof.** Since $\Omega$ is a closed subset of the complete metric space $(X, d)$, the space $(\Omega, d)$ is complete metric as well. Consider the unconstrained mapping $F_\Omega: X \rightrightarrows Z$ defined by

$$F_\Omega(x) := F(x) + \Delta(x; \Omega) \ \text{with} \ \Delta(x; \Omega) := \begin{cases} 0 \in Z & \text{if} \ x \in \Omega, \\ \emptyset & \text{otherwise}. \end{cases} \quad (50)$$
Modify sequentially the mapping $T(x, z)$ from (26) in the proof of Theorem 1 by

$$
T_n(x, z) := \{ y \in X \mid \exists v \in F_\Omega(y) \text{ with } v - z + n^{-1}d(x, y)\xi \leq 0 \}, \ n \in \mathcal{N}. \quad (51)
$$

Fixing $n \in \mathcal{N}$ and following the proof of Theorem 1 with $T_n$ defined in (51), we find a sequence $\{(x_k, z_k)\}$ satisfying

$$
(x_{k+1}, z_{k+1}) \in \text{gph} \ F, \ x_k \in \Omega, \ z_{k+1} - z_k + n^{-1}d(x_{k+1}, x_k)\xi \leq 0 \quad (52)
$$

for all $k = 0, 1, \ldots$. Furthermore, we get $\bar{x} \in \Omega$ (depending on $n \in \mathcal{N}$) by

$$
\bigcap_{k=0}^{\infty} T_n(x_k, z_k) = \{ \bar{x} \} \quad \text{for any fixed } n \in \mathcal{N}. \quad (53)
$$

Since (52) obviously implies that $z_{k+1} \leq z_k$, we find by assumption (48) such $\bar{z} \in \text{Min} F(\bar{x})$ that $\bar{z} \leq z_k$ for all $k = 0, 1, \ldots$. It is not hard to observe, arguing by contradiction and employing (52) and (53) together with the triangle inequality for the metric $d(\cdot, \cdot)$, that

$$
T_n(\bar{x}, \bar{z}) = \{ x \} \quad \text{for all } n \in \mathcal{N}.
$$

Since the pair $(\bar{x}, \bar{z})$ constructed above depends on $n \in \mathcal{N}$, we denote it by $(\bar{x}_n, \bar{z}_n)$ and hence have a sequence $\{(\bar{x}_n, \bar{z}_n)\}$ satisfying

$$
x_n \in \Omega, \ z_n \in F(x_n), \ z_{n+1} \leq z_n, \ T_n(\bar{x}_n, \bar{z}_n) = \{ \bar{x}_n \} \quad (54)
$$

for all $n \in \mathcal{N}$. By the compactness of $\Omega$, we suppose without loss of generality that $x_n \to \bar{x}$ as $n \to \infty$ for some $\bar{x} \in \Omega$. Then, conditions (48) and (54) ensure the existence of $\bar{z}$ satisfying the relationships

$$
\bar{z} \in \text{Min} F(\bar{x}) \quad \text{and} \quad \bar{z} - z_n \in -\Theta \quad \text{for all } n \in \mathcal{N}. \quad (55)
$$

We claim that the pair $(\bar{x}, \bar{z})$ is a weak minimizer to the multiobjective problem (47). Indeed, taking an arbitrary $(x, z) \in \text{gph} \ F$ with $x \in \Omega$ and $(x, z) \neq (\bar{x}, \bar{z})$ and employing (54) and (55), we have by elementary transformations that

$$
z - \bar{z} + n^{-1}d(x_{n+1}, x)\xi \in z_{n+1} - \bar{z} + Z \setminus (-\Theta)
$$

for all $n \in \mathcal{N}$, which easily implies that

$$
z - \bar{z} + n^{-1}d(x_{n+1}, x)\xi \in \Theta + Z \setminus (-\Theta)
$$

and hence $z - \bar{z} + n^{-1}d(x_{n+1}, x)\xi \in Z \setminus (-\Theta)$

due to the convexity of the cone $\Theta$. Now passing to the limit in the last inclusion as $n \to \infty$, we get $z - \bar{z} \in Z \setminus (-\text{int} \ \Theta)$, which justifies the weak minimality of $(\bar{x}, \bar{z})$ to (47).

To complete the proof of assertion (i), it remains to justify that the limiting monotonicity condition (48) is implied by the domination condition (22) of
Theorem 1, where the minimum set $\text{Min}_F(\bar{x})$ is assumed to be closed. Indeed, having the sequence $\{(x_k, z_k)\}$ from the left-hand side of (48), define the sets

$$Q(x_k) := \text{Min}_F(\bar{x}) \cap (z_k - \Theta), \quad k \in \mathbb{N},$$

which are obviously nonempty, closed, and nonincreasing $Q(x_{k+1}) \subset Q(x_k)$ by the monotonicity $z_{k+1} \in z_k - \Theta$ as $k \in \mathbb{N}$ in (48). Hence

$$\bigcap_{k=0}^{\infty} Q(x_k) \neq \emptyset,$$

and any $\bar{z}$ from the above intersection satisfies $\bar{z} \leq z_k$ for all $k \in \mathbb{N}$.

Let us next proceed with the proof of assertion (ii). Having the compact set $\Xi$ from the coercivity condition (49), we consider the auxiliary problem:

$$\text{minimize } F(x) \text{ subject to } x \in \Xi. \tag{56}$$

By assertion (i) of the theorem applied to (56) there is $\bar{x} \in \Xi$ and $\bar{z} \in F(\bar{x})$ such that $(\bar{x}, \bar{z})$ is a weak minimizer to problem (56). We claim that $(\bar{x}, \bar{z})$ is a weak minimizer to $F$ over $\Omega$ as well. Arguing by contradiction, suppose it does not hold and then find $x \notin \Xi$ and $z \in F(x)$ with $z \in \bar{z} - \text{int } \Theta$. By the coercivity condition (49), there are $y \in \Xi$ and $v \in F(y)$ such that $v \leq z$, i.e., $v \in z - \Theta$. The last two inclusions give

$$v \in z - \Theta \subset \bar{z} - \text{int } \Theta - \Theta \subset \bar{z} - \text{int } \Theta,$$

which means that $(\bar{x}, \bar{z})$ is not a weak minimizer to $F$ over $\Xi$. This contradiction completes the proof of (ii) and of the whole theorem.

Note that for scalar cost functions the coercivity condition of Theorem 3(ii) agrees with those from Bao and Khanh (2003), Bianchi, Kassay and Pini (2005), see also the references therein. Observe also that the limiting monotonicity condition (48) in Theorem 3 is strictly better than the domination condition (22) in Theorem 1. We illustrate this by the mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$F(x) = F(x_1, x_2) := \begin{cases} (|x_1|, |x_2|) & \text{if } (x_1, x_2) \neq 0, \\ \mathcal{B} \setminus \{(-1, 0), (0, -1)\} & \text{otherwise.} \end{cases}$$

It is easy to check that the limiting monotonicity condition (48) and relationships in (22) are satisfied, while the minimum set $\text{Min} F(0)$ is not closed.

Our next result establishes the existence of weak minimizers to the constrained multiobjective problem (47) under a new subdifferential extension of the classical Palais-Smale condition to set-valued (and vector-valued) mappings. To formulate this condition, we use the normal subdifferential (16) for set-valued mappings with values in partially ordered spaces introduced in Section 2. Note that new Palais-Smale condition and its application to the proof of the existence
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Theorem rely on the subdifferential variational principle for set-valued mappings established in Theorem 2 and the basic intersection rule for the limiting normal cone (9) derived in Mordukhovich (2006a, Chapter 3).

Recall that the classical Palais-Smale condition for differentiable real-valued function \( \varphi : X \to \mathbb{R} \) asserts that if a sequence \( \{x_k\} \subset X \) is such that \( \{\varphi(x_k)\} \) is bounded and \( \|\nabla \varphi(x_k)\| \to 0 \) as \( k \to \infty \) for the corresponding derivative sequence, then \( \{x_k\} \) contains a convergent subsequence. Our subdifferential extension for set-valued mappings is as follows:

**Definition 3 (subdifferential Palais-Smale condition for set-valued mappings)**

A set-valued mapping \( F : X \to Z \) from a Banach space \( X \) to a partially ordered Banach space \( Z \) with the ordering cone \( \Theta \subset Z \) satisfies the subdifferential Palais-Smale condition if any sequence \( \{x_k\} \subset X \) such that there are \( z_k \in F(x_k) \) and \( x_k^* \in \partial_N F(x_k, z_k) \) with \( \|x_k^*\| \to 0 \) as \( k \to \infty \) (57)

contains a convergent subsequence, where \( \{z_k\} \) is selected to be quasibounded from below.

The subdifferential Palais-Smale condition introduced clearly reduces to the classical one for smooth functions \( F = \varphi \). The next theorem employs the subdifferential Palais-Smale condition to establish the existence of weak minimizers via advanced techniques of variational analysis and generalized differentiation. For simplicity and without loss of generality we consider the (formally) unconstrained case of \( \Omega = X \) in (47). As in the proof of Theorem 3, the general constrained case of (47) can be obviously reduced to the unconstrained one via the restriction \( F_\Omega \) of \( F \) to \( \Omega \) defined in (50).

**Theorem 4 (existence of weak minimizers under the subdifferential Palais-Smale condition)** Let all the assumptions of Theorem 2 be satisfied together with the subdifferential Palais-Smale condition (57). Then \( F \) admits a weak minimizer.

**Proof.** As in the proof of Theorem 3, define the mapping \( T_n : X \times Z \to X \) by (51) with \( F_\Omega = F \) and \( d(x, y) = \|x - y\| \); then construct a sequence \( \{(x_n, z_n)\} \) satisfying relationships (54), where the condition \( x_n \in \Omega \) can be omitted. Following the proof of assertion (i) in Theorem 3, we establish the existence of weak minimizers to \( F \) provided that the above sequence \( \{x_n\} \) contains a convergent subsequence. Let us justify the latter by using the subdifferential Palais-Smale condition of Definition 3, the subdifferential variational principle from Theorem 2, and the basic intersection rule from Mordukhovich (2006a, Theorem 3.4).

To proceed, consider for each \( n \in \mathbb{N} \) the set-valued mapping \( F_n : X \rightrightarrows Z \) given by

\[
F_n(x) := F(x) + g_n(x) \quad \text{with} \quad g_n(x) := n^{-1}\|x - x_n\|\xi
\]

and conclude from (54) and from the structure of \( T_n \) in (51) that \( (x_n, z_n) \) is a strict approximate \( n^{-2}\xi \)-minimizer to \( F_n \). Fix \( n \in \mathbb{N} \) and apply Theorem 2
to $F_n$ and its strict approximate $\varepsilon\zeta$-minimizer $(x_n, z_n)$ with $\varepsilon = n^{-2}$ and $\lambda = n^{-1}$. Taking into account the structure of $F_n$ in (58) and the subdifferential construction (15), we find $(\bar{x}_n, \bar{z}_n, \bar{v}_n, \bar{x}^*_n, \bar{z}^*_n) \in X \times Z \times X^* \times Z^*$ satisfying the relationships

$$z_n \in F(\bar{x}_n), \quad \bar{v}_n = g_n(\bar{x}_n), \quad (\bar{x}_n, \bar{z}_n + \bar{v}_n) \in \text{gph} F_n, \quad \|\bar{x}_n - x_n\| \leq n^{-1},$$

$$\langle \bar{x}^*_n, -\bar{z}^*_n \rangle \in \mathcal{N}\left(\bar{x}_n, \bar{z}_n + \bar{v}_n; \text{epi} F_n\right), \quad -\bar{z}^*_n \in \mathcal{N}(0; \Theta), \quad \|\bar{x}^*_n\| = 1, \quad \|\bar{z}^*_n\| \leq n^{-1}. \quad (59)$$

Define now the following two subsets of the product space $X \times Z \times Z$, which is $\text{epi} F_n$.

$$\Omega_1 := \{(x, z, v) \mid (x, z) \in \text{epi} F\}.$$  

$$\Omega_2 := \{(x, z, v) \mid (x, z) \in X \times Z \mid (x, v) \in \text{epi} g_n\}. \quad (60)$$

It is easy to see that $(\bar{x}_n, \bar{z}_n, \bar{v}_n) \in \Omega_1 \cap \Omega_2$ and both sets $\Omega_i$, $i = 1, 2$, are locally closed around this point by the epiclosedness of $F$ and the Lipschitz continuity of $g_n$. Observe also that the implication

$$(x, z, v) \in \Omega_1 \cap \Omega_2 \Longrightarrow z \in F(x) + \Theta, \quad v \in g_n(x) + \Theta,$$

which ensures therefore that $(x, z + v) \in \text{epi} F_n$. Thus we have from (60) that

$$\limsup_{(x, z, v) \to (\bar{x}_n, \bar{z}_n, \bar{v}_n)} \frac{\langle \bar{x}^*_n, -\bar{z}^*_n \rangle}{\|\bar{x}_n - x_n\|} \leq 0,$$

which implies the inclusions

$$\langle \bar{x}^*_n, -\bar{z}^*_n \rangle \in \mathcal{N}(\bar{x}_n, \bar{z}_n; \Omega_1 \cap \Omega_2) \subset \mathcal{N}(\bar{x}_n, \bar{z}_n; \Omega_1 \cap \Omega_2). \quad (63)$$

Next we are going to express basic normals to the set intersection in (63) via basic normals to $\Omega_1$ and $\Omega_2$ and then—by taking into account the structures of these sets—to arrive at the desired conclusions in terms of the mapping $F$ under consideration. To apply the basic intersection rule from Mordukhovich (2006a, Theorem 3.4) to the intersection $\Omega_1 \cap \Omega_1$, let us first check that the set system $\{\Omega_1, \Omega_2\}$ satisfies the limiting qualification condition at $(\bar{x}_n, \bar{z}_n, \bar{v}_n)$ required in the afore-mentioned theorem. The latter means that for any sequences

$$(x_{ik}, z_{ik}, v_{ik}) \overset{\Omega_1}{\to} (\bar{x}_n, \bar{z}_n, \bar{v}_n) \quad \text{and} \quad (x^*_{ik}, z^*_{ik}, v^*_{ik}) \overset{\omega}{\to} (\bar{x}^*_n, \bar{z}^*_n, \bar{v}^*_n) \quad \text{as} \quad k \to \infty$$

with $(x^*_{ik}, z^*_{ik}, v^*_{ik}) \in \mathcal{N}\left((\bar{x}_n, \bar{z}_n, \bar{v}_n); \Omega_i\right), k \in \mathcal{N}, i = 1, 2$, one has the implication

$$\lim_{k \to \infty} \|\langle x^*_{ik}, v^*_{ik}\rangle + \langle x^*_{2k}, z^*_{2k}, v^*_{2k}\rangle\| \to 0$$

as $k \to \infty \implies (x^*_n, z^*_n, v^*_n) = 0. \quad (64)$
for $i = 1, 2$. To proceed, we observe from the structures of $\Omega_i$ in (61) and (62) that $v_{1k}^* = z_{2k}^* = 0$ for all $k \in \mathcal{N}$, and hence (64) reduces to

$$\left[ \|x_{1k}^* + x_{2k}^*\| \to 0, \|z_{1k}^*\| \to 0, \|v_{2k}^*\| \to 0 \right] \implies x_{1}^* = x_2^* = z_1^* = v^* = 0. \quad (65)$$

Since the conclusions $z_1^* = v_2^* = 0$ are obvious, it remains to show that $x_{1k}^* = x_{2k}^* = 0$.

To this end, observe similarly to the proof of estimate (43) in Theorem 2—based on Mordukhovich (2006a, Theorem 1.43)—that

$$(x_{2k}^*, v_{2k}^*) \in \hat{N}(x_{2k}, g_n(x_{2k})); \text{epi } g_n) \implies \|x_{2k}^*\| \leq n^{-1}\|v_{2k}^*\|$$

for all $k \in \mathcal{N}$, since the mapping $g_n : X \to Z$ from (58) is Lipschitz continuous with modulus $\ell = n^{-1}$. This gives $\|x_{2k}^*\| \to 0$ and hence $\|x_{1k}^*\| \to 0$ as $k \to \infty$ by (65), which justifies the fulfillment of the limiting qualification condition for $\Omega_1, \Omega_2$ at $(\bar{x}_n, \bar{z}_n, \bar{v}_n)$.

To apply the intersection rule from Mordukhovich (2006a, Theorem 3.4), we need also to check that $\Omega_1$ is strongly PSNC at $(\bar{x}_n, \bar{z}_n, \bar{v}_n)$ with respect to the last component $Z$ in the product $X \times Z \times Z$ and that $\Omega_2$ is PSNC at this point with respect to $X \times Z$. The former is obvious from the structure of (61), while the latter follows from (62) due to the Lipschitz continuity of $g_n$; see Mordukhovich (2006a, Corollary 1.69(i)). Thus we have

$$N((\bar{x}_n, \bar{z}_n, \bar{v}_n); \Omega_1 \cap \Omega_2) \subset N((\bar{x}_n, \bar{z}_n, \bar{v}_n); \Omega_1) \cup N((\bar{x}_n, \bar{z}_n, \bar{v}_n); \Omega_2). \quad (66)$$

It follows from (63), (66), and the structures of $\Omega_i$ that there are $u^*_n, p^*_n \in X^*$ satisfying

$$(u^*_n, -z^*_n) \in N((\bar{x}_n, \bar{z}_n); \text{epi } F), \quad (p^*_n, -z^*_n) \in N((\bar{x}_n, g_n(\bar{x}_n)); \text{epi } g_n). \quad (67)$$

and such that $\bar{x}^*_n = u^*_n + p^*_n$. By the condition on $z^*_n$ in (60) and definition (16) of the normal subdifferential we get from (67) the relationships

$$u^*_n \in \partial_N F(\bar{x}_n, \bar{z}_n), \quad p^*_n \in \partial_N g_n(\bar{x}_n), \quad u^*_n + p^*_n = \bar{x}^*_n. \quad (68)$$

It is easy to observe from the form of $g_n$ in (58) with $\|\xi\| = 1$ that $\|p^*_n\| \leq n^{-1}$, and thus—by using the last estimate in (60)—one has

$$\|u^*_n\| = \|\bar{x}^*_n - p^*_n\| \leq \|\bar{x}^*_n\| + \|p^*_n\| \leq n^{-1} + n^{-1} = 2n^{-1}.$$  

Summarizing the above derivation, we have a sequence of triples $\{(\bar{x}_n, \bar{z}_n, u^*_n)\}$ in $X \times Z \times X^*$ satisfying the relationships

$$(\bar{x}_n, \bar{z}_n) \in \text{gph } F, \quad u^*_n \in \partial_N F(\bar{x}_n, \bar{z}_n), \quad \|u^*_n\| \to 0 \text{ as } n \to \infty. \quad (69)$$

Furthermore, the sequence $\{\bar{z}_n\}$ in (69) is quasibounded from below due to this assumption on $F$ induced by Theorem 2. Thus the sequence $\{\bar{x}_n\}$ contains a convergent subsequence as $n \to \infty$ by the subdifferential Palais-Smale condition from Definition 3. Employing the estimate $\|x_n - \bar{x}_n\| \leq n^{-1}$ from (69), we conclude that the initial sequence $\{x_n\}$ selected in the beginning of the proof of this theorem also contains a convergent subsequence. This completes the proof of the theorem.

\[ \square \]
Remark 2 (extensions of the subdifferential Palais-Smale condition and of the weak minimality requirement in the existence theorem) Observe that, by certain modifications of the above proof, we can relax the Palais-Smale condition (57) of Theorem 4 needed for the existence of minimizers and also to relax the notion of minimizers the existence of which is ensured by our approach. Indeed, it is sufficient to replace the normal subdifferential $\partial_N F$ in (57) by its smaller Fréchet counterpart $\hat{\partial}F$ from (15) to proceed in the proof of Theorem 4. To furnish this, we employ in (63) the fuzzy intersection rule from Mordukhovich (2006a, Lemma 3.1)—which is actually equivalent to the extremal principle—instead of the limiting one from Mordukhovich (2006a, Theorem 3.4). It requires, however, more involved arguments. Note to this end that the normal subdifferential form (57) of the Palais-Smale condition from Definition 3 is essentially much more convenient for applications, since our basic subdifferential construction (16) enjoys comprehensive calculus rules (“full calculus”) in contrast to the Fréchet one in (15).

Observe furthermore that the above arguments allows us in fact to ensure the existence of intermediate minimizers (between Pareto (i) and weak Pareto (ii) in Definition 2), which are defined by the replacement of the interior of $\Theta$ in Definition 2(ii) with the relative interior $\text{ri} \Theta \neq \emptyset$ of the ordering cone $\Theta \neq \{0\}$. Indeed, the relationship

$$z - \bar{z} + n^{-1}d(x_{n+1}, x)\xi \in Z \setminus (-\Theta)$$

established in the proof of Theorem 1 implies by passing to the limit as $n \to \infty$ that

$$z - \bar{z} \in Z \setminus (-\text{ri} \Theta).$$

Arguing by contradiction, suppose that

$$z - \bar{z} \not\in Z \setminus (-\text{ri} \Theta), \quad \text{i.e.} \quad z - \bar{z} := \theta \in -\text{ri} \Theta.$$

Hence, there is a number $\eta > 0$ such that

$$(\theta + \eta B) \cap \text{span} \Theta \subset -\text{ri} \Theta.$$  \hfill (71)

We obviously have $n^{-1}d(x_{n+1}, x)\xi \in \eta B$ for all $n \in \mathbb{N}$ sufficiently large. By (71) it gives

$$z - \bar{z} + n^{-1}d(x_{n+1}, x)\xi \in -\text{ri} \Theta,$$

which contradicts (70) and thus justifies the existence of intermediate minimizers to the multiobjective optimization problem (20) under the assumptions of Theorem 4 and its modification mentioned in the first part of this remark.
5. Necessary optimality and suboptimality conditions for constrained multiobjective problems

In the concluding section of the paper we employ the variational principles established in Section 3 and the tools of generalized differentiation from Section 2 to deriving new necessary optimality conditions and suboptimality conditions for general constrained problems of multiobjective optimization. The necessary optimality conditions established below concern minimizers (not just weak minimizers) to multiobjective problems without any interiority requirements imposed on the ordering cone Θ of Z. The (strong) suboptimality conditions are derived in this section for arbitrary approximate εξ-minimizers to multiobjective problems defined by ordering cones with possible empty interiors.

For simplicity we mainly focus in what follows on the class of constrained multiobjective problems given in the form:

$$\text{minimize } f(x) \text{ subject to } x \in \Omega$$  \hspace{1cm} (72)

with a single-valued cost mapping \( f: X \rightarrow Z \) between Asplund spaces and with geometric constraints described by a closed subset \( \Omega \) of \( X \). The results obtained can be extended to more general problems of set-valued optimization with various constraints (of operator, functional, and equilibrium types) based on the extremal and variational principles and on the corresponding generalized differential and SNC calculus rules (the latter calculus is needed only in infinite dimensions)—similarly to the developments and applications in Mordukhovich (2006a,b) for other classes of optimization and equilibrium problems. To illustrate this approach, we present some necessary optimality and suboptimality conditions derived in this way for multiobjective problems with functional constraints given by finitely many equalities and inequalities via (generally non-smooth) real-valued functions.

Let us start with necessary optimality conditions for local minimizers to problem (72), where an optimal solution (minimizer) is understood in the sense of Definition 2(i) with the usual neighborhood localization. Recall that the corresponding subdifferential constructions and SNC properties used in the theorem are defined and discussed in Section 2.

**Theorem 5** (necessary optimality conditions for multiobjective problems with geometric constraints) Let \( \bar{x} \) be a local minimizer to problem (72) with \( \bar{z} := f(\bar{x}) \), where the ordering cone \( \Theta \subset Z \) satisfies the standing convexity, closedness and pointedness assumptions, where \( f \) is locally epiclosed around \((\bar{x}, \bar{z})\), and where \( \Omega \) is locally closed around \( \bar{x} \). Suppose also that \( \Theta \) is SNC at the origin, that either \( \Theta \) is SNC at \( \bar{x} \) or \( f \) is partially SNEC at \((\bar{x}, \bar{z})\), and that the qualification condition

$$\partial^\infty f(\bar{x}) \cap (-N(\bar{x};\Omega)) = \{0\}$$  \hspace{1cm} (73)

is satisfied. Then one has the inclusion

$$0 \in \partial_N f(\bar{x}) + N(\bar{x};\Omega).$$  \hspace{1cm} (74)
Proof. Consider the restriction $f_\Omega$ of the mapping $f$ to the set $\Omega$ given by

$$f_\Omega(x) := f(x) + \Delta(x; \Omega),$$

(75)

where the indicator mapping $\Delta(\cdot; \Omega)$ of $\Omega$ is defined in (50). Taking any $\xi \in \Theta$ with $\|\xi\| = 1$ and any $k \in \mathbb{N}$, observe that $(\bar{x}, \bar{z})$ is a (local) strict approximate $k^{-1}\xi$-minimizer to $f_\Omega$ in the sense of Definition 2(iv). It is easy to see that $f_\Omega$ satisfies (locally) all the assumptions required by Theorem 2 except that of $\xi \in \text{int } \Theta$, which is not needed in what follows. Employing the latter theorem and relationships (40) of the extremal principle in its proof, we find sequences $\{(x_k, x_k^*)\} \in \Omega \times X^*$ such that

$$x_k^* \in \mathcal{D} f_\Omega(x_k), \quad \|x_k - \bar{x}\| \leq k^{-1}, \quad \text{and} \quad \|x_k^*\| \leq k^{-1} \text{ for all } k \in \mathbb{N}. \quad (76)$$

By definition of the Fréchet subdifferential in (15) with $\varepsilon = 0$, for each $k \in \mathbb{N}$ we find $z_k \in f(x_k) + \Theta$ and $-z_k^* \in N(0; \Theta)$ with $\|z_k^*\| = 1$ such that

$$x_k^* \in \mathcal{D}^* \mathcal{E}_{f_\Omega}(x_k, z_k)(z_k^*) \text{ for all } k \in \mathbb{N}. \quad (77)$$

Since the unit ball $\mathcal{B}^*$ of $Z^*$ is sequentially weak$^*$ compact, we select a subsequence of $\{z_k^*\}$ that weak$^*$ converges to some $z^* \in \mathcal{B}^*$. Note that $z^* \neq 0$, because the converse property implies that $z_k^* \rightharpoonup^* 0$ and hence $\|z_k^*\| \to 0$ as $k \to \infty$ by the assumed SNC property of $\Theta$. Furthermore, by $\|x_k^*\| \to 0$ in (76) we may assume without loss of generality that $\|z^*\| = 1$, since otherwise we can normalize the inclusion in (77) and keep the convergence $\|x_k^*\| \to 0$. Passing to the limit in (77) as $k \to \infty$ and taking into account the structure of the restriction $f_\Omega$ in (75), we get

$$0 \in \partial_N f_\Omega(\bar{x}) = \partial_N \left[ f + \Delta(\cdot; \Omega) \right](\bar{x}). \quad (78)$$

It follows from (16) and (78) that

$$(0, -z^*) \in N((\bar{x}, \bar{z}); \text{epi } f \cap (\Omega \times Z)) \text{ for some } -z^* \in N(0; \Theta), \quad \|z^*\| = 1. \quad (79)$$

Employing now in (79) the basic intersection rule from Mordukhovich (2006a, Theorem 3.4) whose requirements are satisfied due to the qualification condition (73) and the SNC assumptions of this theorem, we get from (79) that

$$(0, -z^*) \in N((\bar{x}, \bar{z}); \text{epi } f) + N(\bar{x}; \Omega) \times \{0\} \text{ with } -z^* \in N(0; \Theta), \quad \|z^*\| = 1,$$

which is obviously equivalent to (74). This completes the proof of the theorem. \hfill \blacksquare

It occurs that the qualification condition (73) and the partial SNEC condition of Theorem 5 are automatically fulfilled for a major class of epi-Lipschitz-like (ELL) cost mappings $f : X \to Z$ described in Section 2.
Corollary 2 (necessary optimality conditions for multiobjective problems with Lipschitzian costs) Let \( \bar{x} \) be a local minimizer to (72), where the ordering cone \( \Theta \) satisfies the assumptions of Theorem 5, where the constraint set \( \Omega \) is locally closed around \( \bar{x} \), and where the cost mapping \( f \) is epiclosed and \( ELL \) around \( (\bar{x}, \bar{z}) \) with \( \bar{z} = f(\bar{x}) \). Then the necessary optimality condition (74) is satisfied.

**Proof.** This follows from Theorem 5 due to Proposition 1, ensuring simultaneously the partial SNEC property and the qualification condition (73) for \( ELL \) mappings.

Next we present a specification of Theorem 5 for multiobjective problems (72) with functional constraints given in the conventional form of mathematical programming:

\[
\Omega := \{ x \in X \mid \varphi_i(x) \leq 0, \ i = 1, \ldots, m; \ \varphi_i(x) = 0, \ i = m + 1, \ldots, m + r \}. \tag{80}
\]

For simplicity we assume that all the functions \( \varphi_i : X \to \mathbb{R} \) are locally Lipschitzian around the reference point; more general non-Lipschitzian settings can be also considered based on the calculus rules of Mordukhovich (2006a).

The following consequence of Theorem 5 holds.

**Corollary 3** (necessary optimality conditions in multiobjective mathematical programming) Let \( \bar{x} \) be a local minimizer to problem (72) with the constraint set \( \Omega \) given by (80), where the ordering cone \( \Theta \) satisfies the assumptions of Theorem 5, where the cost mapping \( f \) is epiclosed around \( (\bar{x}, \bar{z}) \) with \( \bar{z} = f(\bar{x}) \), and where all the functions \( \varphi_i \) are locally Lipschitzian around \( \bar{x} \). Impose the two qualifications conditions

\[
[0 \in \sum_{i=1}^{m} \lambda_i \partial \varphi_i(\bar{x}) + \sum_{i=m+1}^{m+r} \lambda_i (\partial \varphi_i(\bar{x}) \cup \partial (-\varphi_i)(\bar{x})), \\
\lambda_i \geq 0 \ \text{for} \ i = 1, \ldots, m + r, \ \lambda_i \varphi_i(\bar{x}) = 0 \ \text{for} \ i = 1, \ldots, m] \\
\Rightarrow \lambda_i = 0 \ \text{for all} \ i = 1, \ldots, m + r;
\]

\[
[-\partial^\infty f(\bar{x}) \ni -x^* \in \sum_{i=1}^{m} \lambda_i \partial \varphi_i(\bar{x}) + \sum_{i=m+1}^{m+r} \lambda_i (\partial \varphi_i(\bar{x}) \cup \partial (-\varphi_i)(\bar{x})) \] \quad \text{with} \\
\lambda_i \geq 0 \ \text{as} \ i = 1, \ldots, m, \ \lambda_i \varphi_i(\bar{x}) = 0 \ \text{as} \ i = 1, \ldots, m] \Rightarrow x^* = 0 \tag{82}
\]

formulated via the basic subdifferential, Mordukhovich (2006a), of Lipschitzian functions \( \varphi_i \). Then there are \( \lambda_i \geq 0 \) for \( i = 1, \ldots, m + r \) such that \( \lambda_i \varphi_i(\bar{x}) = 0 \) as \( i = 1, \ldots, m \) and

\[
0 \in \partial_N f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial \varphi_i(\bar{x}) + \sum_{i=m+1}^{m+r} \lambda_i (\partial \varphi_i(\bar{x}) \cup \partial (-\varphi_i)(\bar{x})). \tag{83}
\]
Proof. First observe that the basic normal cone \( N(\cdot; \Omega) \) to the constraint set \( \Omega \) given in (80) satisfies the inclusion
\[
N(\bar{x}; \Omega) \subset \left\{ \sum_{i=1}^{m} \lambda_i \partial \varphi_i(\bar{x}) + \sum_{i=m+1}^{m+r} \lambda_i (\partial \varphi_i(\bar{x}) \cup \partial (\varphi_i)(\bar{x})) \right\} \tag{84}
\]
provided the fulfillment of the qualification condition (81); see, e.g., Mordukhovich (2006a, Corollary 4.36). Substituting (84) into (73) and (74), we get the qualification condition (82) and optimality condition (83), respectively. Finally, the qualification condition (81) ensures the SNC property of the constraint set (80) at \( \bar{x} \); this follows from Mordukhovich (2006a, Theorem 3.86). Thus we meet all the requirements of Theorem 5 and complete the proof of the corollary.

Note that the qualification condition (81) reduces to the classical Mangasarian-Fromovitz constraint qualification when the functions \( \varphi_i \) are strictly differentiable at \( \bar{x} \) (in particular, when \( \varphi_i \in C^1 \) around \( \bar{x} \)); in this case \( \partial \varphi(\bar{x}) = \{\nabla \varphi(\bar{x})\} \). Note, furthermore that, by Corollary 2, the qualification condition (82) is automatic if the cost mapping \( f \) is ELL around \( \bar{x} \). For the latter class we also have the partial SNEC property of \( f \) at \( (\bar{x}, \bar{z}) \), which is not needed in the framework of Corollary 3 under the generalized Mangasarian-Fromovitz constraint qualification (81).

Our final result concerns suboptimality conditions for problem (72) applied to its approximate solutions—the exact minimizers may not even exist.

**Theorem 6 (suboptimality conditions in multiobjective optimization)** Let \( \bar{x} \) be a local approximate \( \varepsilon \xi \)-minimizer to problem (72) in the sense of Definition 2(ii) with \( \varepsilon > 0 \) and \( 0 \neq \xi \in \Theta \), let \( \lambda > 0 \), and let the ordering cone \( \Theta \subset Z \) satisfy the requirements of Theorem 5. Suppose furthermore that for any approximate \( \varepsilon \xi \)-minimizer \( x \in \Omega \cap (\bar{x} + \eta I B) \) with some \( \eta > \lambda \) and with \( z := f(x) \leq f(\bar{x}) =: \bar{z} \) the following assumptions hold:

- \( \Omega \) is locally closed around \( x \) and \( f \) is epiclosed around \( (x, z) \);
- either \( \Omega \) is SNC at \( x \), or \( f \) is partially SNEC at \( (x, z) \);
- one has the qualification condition
\[
\partial^\infty f(x) \cap (-N(x; \Omega)) = \{0\}. \tag{85}
\]

Then there is a local approximate \( \varepsilon \xi \)-minimizer \( \hat{x} \in \Omega \) to problem (72) with \( \|\hat{x} - \bar{x}\| \leq \lambda \) and \( f(\hat{x}) \leq f(\bar{x}) \) satisfying the suboptimality relationships
\[
\|\hat{x}_f^* + \hat{x}_\Omega^*\| \leq \frac{\varepsilon}{\lambda} \text{ for some } \hat{x}_f^* \in \partial N f(\hat{x}) \text{ and } \hat{x}_\Omega^* \in N(\hat{x}; \Omega). \tag{86}
\]
Proof. Employing Corollary 1 with $x_0 := \tilde{x}$ to the restricted mapping $f_\Omega$ in (75), we find $\hat{x} \in \Omega \cap (\tilde{x} + \lambda \mathcal{B})$ with $f(\hat{x}) \leq f(\tilde{x})$, which is obviously a local approximate $\varepsilon$-minimizer to $f$ on $\Omega$; furthermore, it provides an exact local minimum to the perturbed mapping

$$g(x) := f(x) + \frac{\varepsilon}{\lambda} \|x - \tilde{x}\| \xi \quad \text{over} \quad x \in \Omega.$$  

(87)

Applying now Theorem 5 to (87), we get the optimality condition

$$0 \in \partial_N g(\tilde{x}) + N(\tilde{x}, \Omega)$$  

(88)

under the assumptions of the latter theorem imposed on $g$. It follows from definition (17) of the singular subdifferential, the Lipschitz continuity of the perturbation in (87), and the mixed coderivative sum rule from Mordukhovich (2006a, Theorem 3.10) that $\partial^\infty g(\tilde{x}) = \partial^\infty f(\tilde{x})$, and thus the qualification condition (73) for $g$ is equivalent to (85) at $x = \hat{x}$. Taking into account the SNC calculus result of Mordukhovich (2006a, Theorem 3.88), we easily conclude from (87) that the SNEC requirement on $g$ agrees with that on $f$ at $\tilde{x}$. Finally, it follows from the normal subdifferential construction (16) and from the normal coderivative sum rule in Mordukhovich (2006a, Theorem 3.10) that

$$\partial_N g(\tilde{x}) \subset \partial_N f(\tilde{x}) + \varepsilon \lambda \mathcal{B}^*.$$  

(89)

Substituting (89) into (88), we arrive at the suboptimality relationships in (86) and thus finish the proof of theorem.

\[\fbox{\begin{array}{c} \text{Similarly to Corollaries 2 and 3, we can establish the corresponding consequences of Theorem 6 that provide suboptimality conditions to multiobjective problems with Lipschitzian costs and with functional constraints.} \\
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\text{References} \\


