This dissertation is written in respond to the increasing needs for studying set-valued optimization by using advanced tools of variational analysis.

There is no doubt that the discovery of the derivative by Newton and Leibnitz, mainly motivated by the developments of Fermat in the second half of the seventeenth century, was one of the most important events in the history of mathematics. Differential and Integral Calculus laid the foundation for the rapid development of literally every area of applied and fundamental sciences. For more than two centuries, this tool was used to solve all kinds of optimization problems. However, it gradually became clear that the challenges science started to face could not be resolved with the help of Differential and Integral Calculus only. One of the first mathematicians who formulated and successfully solved a nonsmooth optimization problem was Chebyshev. He considered a problem of best approximation of a function by a polynomial which had to be constructed in the sup-norm, which is essentially nonsmooth. Nonsmooth problems continued to pop out from various areas of science, data and ideas continued to accumulate for the following centuries and finally led to the rapid development of Nonsmooth Analysis. It started with Convex Analysis [20] and then further extended to Nonconvex Analysis [9] and Variational Analysis [21, 8, 16].

Vector optimization has its root in economic equilibrium and welfare theories. Edgeworth (1881) and Pareto (1896) are credited for first introducing the concept of noninferiority (known as Pareto
Abstract

minimality) in economics. Given a set of alternative allocations, says, goods or income for a set of individuals, an allocation is Pareto optimal if no alternative can make at least one individual better off without making any other individual worse off. Since then, it has permeated many areas of science, engineering, economics, management, and operations research. During the last three decades vector optimization has been further extended to the realm of set-valued optimization, see [14, Chapter 5]. This new field of research seems to have numerous important applications in stochastic programming, fuzzy programming and optimal control, etc. In particular, in the navigation and control of autonomous transportation robots one uses ultrasonic sensors determining the smallest distance to an obstacle in the emission cone. Since the direction of the object cannot be identified in this cone, the location of the object is set-valued. Therefore, it may leads to problems of set-valued optimization. In [14] set-valued optimization was systematically investigated by using the primal approach via contingent derivatives and epiderivatives of set-valued mappings. However, this approach seems to be not efficient since these generalized differentiations do not enjoys a rich calculus.

The principal theme of this dissertation is the development of new techniques of variational analysis to study multiobjective/set-valued optimization. In particular, we develop two new variational principles which are extensions of Ekeland variational principle and subdifferential variational principle to set-valued mappings to investigate conditions ensuring existence of optimal solutions, necessary conditions for optimal solutions (if exist), and suboptimality necessary conditions in set-valued optimization. In contrast to conventional scalarization methods, which convert a multiobjective optimization problem to a scalar one so that one can apply known results to the latter problem to understand the former one, our dual approach deals directly with multiobjective optimization problems by using advanced tools of variational analysis and generalized differentiation. In addition, we introduce new constructions of generalized differentiation, namely, subdifferentials of set-valued mappings.
This dissertation is the study of an important class of constrained optimization, namely multi-objective optimization problems with equilibrium constraints (MOPEC) given by

$$\text{minimize} \quad F(x, y) \quad \text{subject to} \quad x \in \Omega, \quad y \in S(x),$$  

(1)

where $F : X \times Y \rightarrow Z$ is a set-valued cost mapping, $\Omega$ is a subset of $X$, $S : X \rightarrow Y$ is a set-valued mapping describing the so-called equilibrium constraints and governed by extended equations

$$0 \in G(x, y) + Q(x, y),$$  

(2)

where the base $G : X \times Y \rightarrow W$ and the field $Q : X \times Y \rightarrow W$ are set-valued mappings between Banach spaces. MOPEC (1) contains almost all models of mathematical programs in optimization.

It has three important features which distinguish itself from the standard mathematical programs with equilibrium constraints, which cover a variety of models particularly arising in applications to problems of operations research, engineering, economics, Stackelberge game, etc, see \[12, 15, 17\].

Firstly, the cost mapping might have set-valued values. It not only extends the domain of applications to problems of set-valued optimization but also opens new insights for some classical problems and thus might lead to more efficient approaches to solve and resolve them. Let us illustrate this with two important problems. One is the class of bilevel mathematical programs.

The upper level problem is

$$\text{minimize} \quad \varphi(x, y) \quad \text{subject to} \quad x \in \Omega, \quad y \in S(x),$$  

(3)

where $\varphi : X \times Y \rightarrow [0, \infty), \Omega \subset X$, and $M(x)$ is the solution set of the lower-level problem

$$\text{minimize} \quad \psi(x, y) \quad \text{subject to} \quad y \in \Xi(x),$$  

(4)

where $\psi : X \times Y \rightarrow [0, \infty)$, and $\Xi(x) \subset Y$ is a set depending on $x$. Obviously, the term “minimization” in the upper level problem is slightly ambiguous in the classical minimization sense unless the lower level problem has a unique solution. However, the primary complication in bilevel programming
Abstract

Bao Truong

arises when the lower level problem does not have a unique solution. In set-valued optimization, the two-level problem (3)-(4) is described by

\[
\begin{align*}
\text{minimize} & \quad F(x) \quad \text{subject to} \quad x \in \Omega, \\
\end{align*}
\]

where \( F : X \to \mathbb{R} \) is a set-valued mapping with images \( F(x) := \cup \{ \varphi(x, y) \mid y \in M(x) \} \). The other example is a general mathematics model of welfare economics. Let \( E \) be a normed commodity space of an economy \( \mathcal{E} = (C_1, \ldots, C_n, S_1, \ldots, S_m, W) \) with \( n \in \mathbb{N} \) customers and \( m \in \mathbb{N} \) firms/producers, where \( C_i \subset E \) stands for the consumption set of customer \( i \) for \( i = 1, \ldots, n \), \( S_j \) denotes the production set of firm \( j \) for \( j = 1, \ldots, m \), and \( W \) is a net demand constraint set representing constraints related to the initial inventory (scarce resources) of commodities in the economy \( \mathcal{E} \).

Note that the net demand constraint set \( W \) not only allows us to unify some common situations in economic modeling; in particular, the exchange economic model with \( W := \{0\} \), the markets clear model with \( W = \{ \bar{w} \} \) being the aggregate endowment of scarce resources of \( \mathcal{E} \), the model of implicit free disposal of commodities with \( W = \bar{w} + E_+ \), where \( E_+ \) denoting the positive cone of the commodity space \( E \), etc. but also reflects possible uncertainties in the economic model \( \mathcal{E} \).

Each customer has his own preference \( \prec_i \) reflecting the objective, taste or feeling of customer \( i \) for all \( i = 1, \ldots, n \). Denoting a production strategy by \( y := (y_1, \ldots, y_m) \in S_1 \times \ldots \times S_m \) and a consumption plan by \( z := (z_1, \ldots, z_n) \in C_1 \times \ldots \times C_n \), the pair \((y, z)\) is a feasible allocation of the economy \( \mathcal{E} \) if it satisfies the markets constraint condition \( w := \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \in W \). A feasible allocation \((\bar{y}, \bar{z})\) is weak Pareto optimal if there does not exist any feasible allocation \((y, z)\) such that \( z_i \) is referred to \( \bar{z}_i \) by customer \( i \) for all \( i = 1, \ldots, n \). It is shown in [7] that each Pareto optimal allocation of \( \mathcal{E} \) is a minimizer of the corresponding constrained set-valued optimization problem

\[
\begin{align*}
\begin{cases}
\text{minimize} & \quad F(x) := \left\{ z \in Z \mid w = \sum_{i=1}^{n} z_i - \sum_{j=1}^{m} y_j \right\} \\
\text{subject to} & \quad x \in \Omega := \left( \prod_{j=1}^{m} S_j \right) \times W \subset X, \\
\end{cases}
\end{align*}
\]

where “minimization” in (5) is understood with respect to the product preference \( \prec := \prod_{i=1}^{n} \prec_i \) on \( Z \).
defined by $z \prec u$ if and only if $z_i \prec_i u_i$ for all $i = 1, \ldots, n$.

Secondly, MOPEC (1) consists of the presence of the “equilibrium constraint” mapping described by an extended equation (2) with both set-valued mappings $G$ and $Q$. It has been well recognized, starting with the seminal work by Robinson [19], that solution sets to optimization-related problems arising in both the theory and applications can be conveniently described by the so-called parameterized generalized equations in the form

$$0 \in g(x, y) + Q(y)$$

with the decision variable $y \in Y$ and the parameter $x \in X$, where $g: X \times Y \to W$ is a single-valued mapping while $Q: Y \to W$ is a set-valued one. In particular, the generalized equation (6) reduces to the parametric variational inequality when $Q(y) = N(y; \Xi)$ is the normal cone mapping to a convex set, to the classical parametric complementarity problem when $\Xi$ is the nonnegative orthant in $\mathbb{R}^n$.

It is well known that the latter model covers sets of optimal solutions with the associated Lagrange multipliers or sets of KKT vectors satisfying first-order necessary optimality conditions in parametric problems of nonlinear programming with smooth data. Note that the classical generalized equation model (6) does not cover nevertheless some classes of problems important in applications.

Let us particularly mention the following two equilibrium constraints $S(\cdot)$ governed by extended equations (2):

- $S(x)$ stand for sets of solutions to the so-called set-valued variational inequality (SVI): given $G: X \times Y \rightrightarrows Y^*$ and $\Xi \subset Y$, find $y \in \Xi$ such that

$$\text{there is } y^* \in G(x, y) \text{ with } \langle y^*, u - y \rangle \geq 0 \text{ for all } u \in \Xi. \quad (7)$$

Obviously, SVI (7) reduces to the standard variational inequality in [11] when the mapping $G = g: X \times Y \to Y^*$ is single-valued.

- Consider the parametric problem of nonsmooth constrained optimization:

$$\text{minimize } \psi(x, y) \text{ subject to } y \in \Xi \subset Y, \quad (8)$$
where $\psi: X \times Y \to \mathbb{R}$ is a lower semicontinuous function and $\Xi$ is a closed set. Then first-order necessary optimality conditions for (8) are written in the form

$$0 \in \partial_y \psi(x, y) + N(y; \Xi)$$

via appropriate subdifferentials of $\varphi$ with respect to $y$ and normal cones to $\Xi$. Moreover, the optimality condition (9) in the convex case is known to be not only necessary but also sufficient.

In the latter case, the problem of minimizing a scalar function $\varphi(x, y)$ subject to (9) reduces to a nondifferentiable bilevel program; see \[10\]. Since the subdifferential mapping $\partial_y \varphi(x, y)$ is always set-valued unless $\varphi$ is smooth, our extended equation is the best fitted.

Thirdly, the “minimality” is understood in the sense of Pareto optimality in infinite-dimensional partially ordered image spaces. It is worth remarking that one of the most common techniques wildly used in vector optimization is scalarization that requires the nonempty interiority property of the ordering cone in both convex and nonconvex optimization; see \[13\]. But “the class of ordered topological vector spaces possessing cones with nonempty interiors is not very broad” (see \[18\]); in particular, the natural ordering cones in the Lebesgue spaces $l^p$ and $L^p$ for $1 \leq p < \infty$ have empty interior. Therefore, such a condition is quite restrictive and will not require in our model. In this dissertation, we further develop the variational approach first realized in \[17\] and strongly based on the extremal principle that can be treated as a variational counterpart of the classical separation in the case of nonconvex sets. This approach is considered as the dual approach in comparison with the primal approach in Jahn’s book \[14\].

This dissertation contains new results of the author and his advisor at Wayne State University published in six papers \[1–6\]. It is organized as follows.

Chapter 1 presents strong motivations of the class of MOPECs (1) and the new approach used to study these problems.

In Chapter 2, we briefly review certain basic tools of variational analysis and generalized dif-
Abstract

Bao Truong

ferentiation widely used in the dissertation. After recalling basic constructions of normal cones to sets and coderivatives to set-valued mappings, we introduce new subdifferential notions for set-valued mappings (in particular, for vector-valued mappings) with values in partially ordered spaces. They can be viewed as extensions of Mordukhovich nonconvex basic and singular subdifferentials of extended-real-valued functions. It is important to emphasize that they inherit the full calculus for coderivatives of set-valued mappings developed in [ 17, Chapter 3 ]. Moreover, several important relationships between coderivatives and subdifferentials of set-valued mappings, which are even new for extended-real-valued functions, are studied. Subdifferentials of set-valued mappings are first introduced in [ 2 ], then further developed in [ 3–6 ].

In Chapter 3 we develop a new variational tool to investigate conditions ensuring the existence of optimal solutions to constrained/unconstrained multiobjective optimization problems. Our primary aim is an extension of subdifferential variational principle to set-valued mappings, which is a variational counterpart of local separation for nonconvex sets and it was established by Mordukhovich and Wang (see [ 16, Theorem 2.28 ]) and based on the extremal principle as well as the celebrate Ekeland variational principle. The later principle was first introduced by Ekeland in [ 11 ] and soon became a powerful tool in various fields such as nonlinear analysis, variational analysis and optimization, etc. with numerous generalizations and equivalent formulations. Since none of generalizations of the Ekeland variational principle for set-valued meet our purposes, we first formulate a new version of the seminal derivative-free Ekeland variational principle for set-valued mappings and then use it in the proof of the new subdifferential variational principle for set-valued mappings. Both set-valued variational principles presented in this chapter were first introduced in [ 2 ], then further improved in [ 6 ].

Chapter 4 contains applications of the variational techniques and principles developed in Chapter 3 to deriving efficient conditions for the existence of optimal solutions to both constrained and unconstrained multiobjective optimization problems. In particular, we establish new conditions
of the coercivity type and of the Fréchet subdifferential Palais-Smale type for set-valued and non-smooth single-valued mappings ensuring the existence of weak minimizers to the multiobjective optimization problems under consideration. Since the Fréchet subdifferentials, which are sizeably smaller than their corresponding limiting counterparts but do not enjoy good calculus, the Fréchet Palais-Smale condition is replaced by the corresponding limiting one in order to seek conditions ensuring the existence of optimal solutions to MOPEC (1). The results in this chapter are new in both finite-dimensional and infinite-dimensional spaces and they can be found in [2, 6].

Chapter 5 is devoted to applications of the extremal principle and basic calculus rules for generalized differentiation to deriving necessary optimality conditions for Pareto-type minimizers to constrained/unconstrained multiobjective optimization problems containing geometric constraints and extended equilibrium constraints. First we provides the generalized Fermat rules for set-valued mappings from the dual-space approach. It bases on the extremal principle of variational analysis and calculus rules of generalized differentiation. The reader can find other proofs in [2, Theorem 5] and [6, Theorem 5.1] which rely on the subdifferential variational principle instead of the extremal principle. We clearly point out that the generalized Fermat rule with Fréchet subdifferentials of costs mappings is a characterization of scalar optimization, and that it no longer holds in set-valued optimization. Then, we derive necessary optimality conditions for MOPEC (1) by introducing a new auxiliary problem to MOPEC (1). It allows us to weaken requirements imposed on data of problems under consideration. The necessary optimality conditions obtained in Section 5, being new in finite and infinite dimensions, unify and improve various results in single-objective and vector-objective optimization with equality, inequality, operator, and other type of constraints known in the literature; see the discussion in Remark 4.4 in [3].

The concluding Chapter 6 is devoted to considering a specification of MOPECs with only one objective, known as MPECs. In contrast to Chapter 5 we study weak and strong suboptimality conditions for MPECs. The significant advantage of these suboptimality conditions do not assume
the existence of optimal solutions and are important for both theoretical and numerical aspects of multiobjective optimization. The results of this section were in [4].

References


