NMU Math & CS Department

Problem of the Month, November 2024

Consider the strange sequence of functions $F_1, F_2, F_3, F_4 \ldots$ given by:

$$
\begin{aligned}
&F_1(a) = a \\
&F_2(a, b) = a + b \\
&F_3(a, b, c) = (a + b, a + c, b + c) \\
&F_4(a, b, c, d) = (a + b, a + c, a + d, b + c, b + d, c + d) \\
&\qquad \vdots \\
&F_n(z_1, z_2, \ldots, z_n) = (z_1 + z_2, z_1 + z_3, \ldots, z_1 + z_n, z_2 + z_3, \ldots, z_{n-1} + z_n).
\end{aligned}
$$

The first thing you may notice is that for $n \geq 2$, if there are n coordinates on the left side, there will be $(n-1) + (n-2) + \ldots + 1 = \frac{n(n-1)}{2}$ coordinates on the right side. Therefore you may interpret these functions as mappings between tuples of integers, and we will in fact consider them as mappings between **tuples of positive integers**.

$$F_n : \mathbb{N}^n \longrightarrow \mathbb{N}^{n(n-1)/2}, \text{ for } n \geq 2.$$

For example we have $F_3 : \mathbb{N}^3 \to \mathbb{N}^3$, and $F_4 : \mathbb{N}^4 \to \mathbb{N}^6$, and $F_4(1, 2, 3, 4) = (3, 4, 5, 5, 6, 7)$.

The Warm up. Show that if $X, Y \in \mathbb{N}^n$ are two n-tuples with "the same elements" up to reordering, then $F_n(X)$ and $F_n(Y)$ have "the same elements" up to reordering. As an example when $n = 3$: $(1, 2, 3)$ and $(2, 3, 1)$ have the same elements up to reordering, and we see that $F_3(1, 2, 3) = (3, 4, 5)$ and $F_3(2, 3, 1) = (5, 3, 4)$ have the same elements up to reordering.

The Problem. Show that the following statement is true precisely when $n \neq 2^k$.

$(F_n(X), F_n(Y)$ same elements up to reordering$) \Rightarrow (X, Y$ same elements up to reordering$)$

Hint: Given any tuple $W = (a_1, \ldots, a_k)$, try to make use of the associated polynomial $p_W(x) = x^{a_1} + \cdots + x^{a_k}$. For example $p_{(1,2,3)}(x) = x + x^2 + x^3$.

The NMU Mathematics and Computer Science Department invites you to participate in the 2024/2025 Problem of the Month contest to have some fun and get a little recognition. There are paper copies of the problems available at the department front desk if you'd like.

Rules: Anyone is welcome to submit a solution, and all correct solutions will be recognized; but only undergraduates enrolled in coursework at NMU are eligible to win the prize at the end of the year. The first student to submit a correct solution is the winner of the month. The top problem solver for the academic year will receive a fabulous prize, and be recognized at the department year-end celebration. You must write clear and complete solutions to the problems. You must include your name, NMU IN, address, phone, email, and exact date/time of your submission. You may either submit in person at the department front desk (use a staple if there are multiple pages), or email your solution in `pdf` format to `darowe@nmu.edu`.