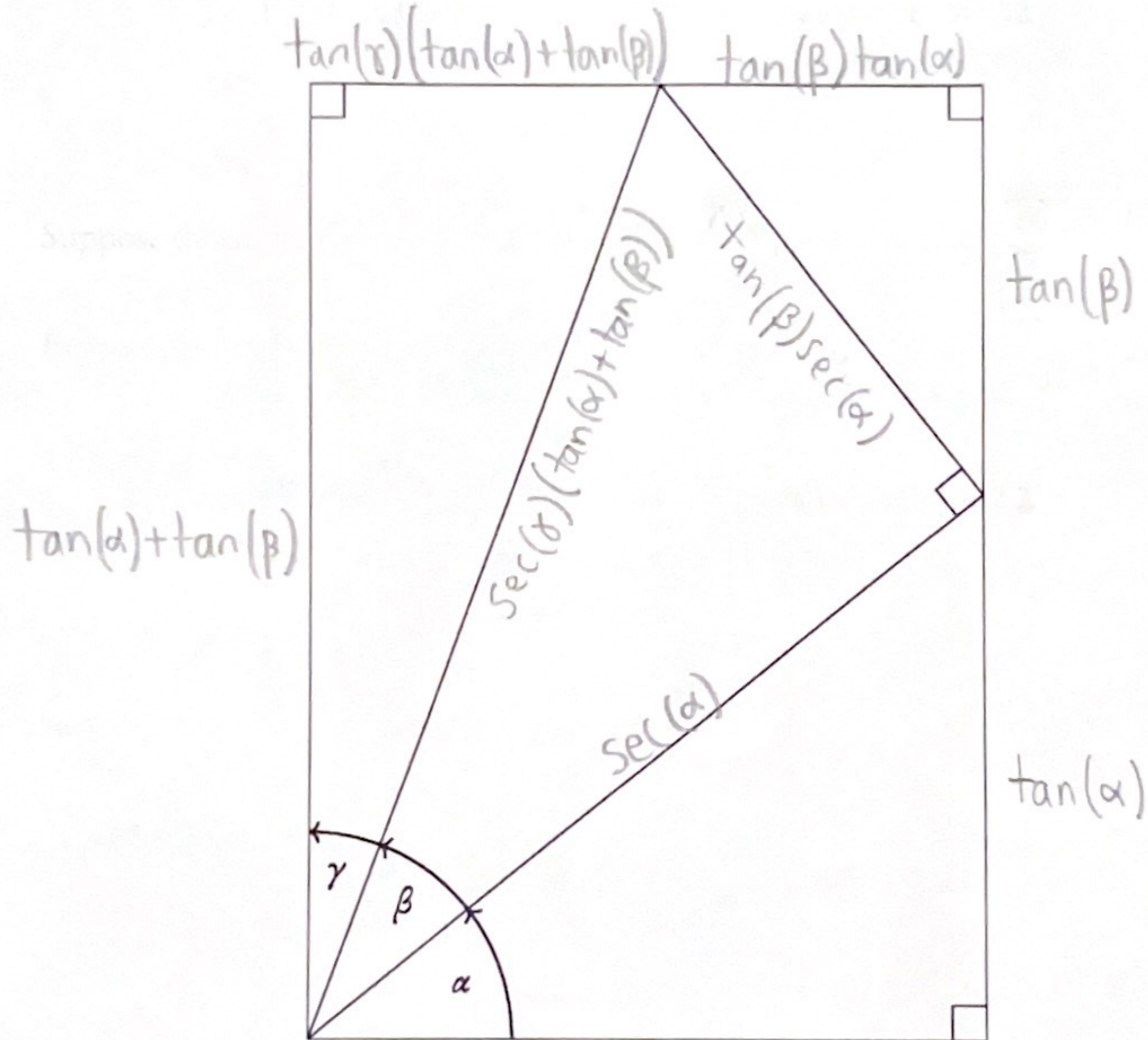# NMU Math & CS Department

Problem of the Month, October 2022

By Daniel Rowe

Suppose that α, β, γ are positive angles with $\alpha + \beta + \gamma = \dfrac{\pi}{2}$.

Express the unknown lengths in the picture below in such a manner that proves:

$$\tan(\alpha)\tan(\beta) + \tan(\beta)\tan(\gamma) + \tan(\alpha)\tan(\gamma) = 1.$$

Diagram labels:
- Top-left segment: $\tan(\gamma)\left(\tan(\alpha)+\tan(\beta)\right)$
- Top-right segment: $\tan(\beta)\tan(\alpha)$
- $\tan(\beta)\sec(\alpha)$
- $\sec(\gamma)\left(\tan(\alpha)+\tan(\beta)\right)$
- Left side: $\tan(\alpha)+\tan(\beta)$
- $\tan(\beta)$
- $\sec(\alpha)$
- $\tan(\alpha)$
- Angles at bottom-left vertex: γ, β, α

By Daniel Rowe

Suppose that $X \in \mathbb{C}$ is a number such that:

$$X^3 + \frac{1}{X^3} = \sqrt{2}.$$

Find the value of the expression:

$$X^{2022} + \frac{1}{X^{2022}}.$$

__Solution #1__: Let $Y = X^3$ then $Y + Y^{-1} = \sqrt{2}$ and $Y \cdot Y^{-1} = 1$

Thus Y is a root of $t^2 - \sqrt{2}t + 1 = 0$,

i.e. $Y = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i = e^{i\left(\frac{\pi}{4}\right)}$ OR $e^{i\left(-\frac{\pi}{4}\right)}$.

Since Y satisfies $Y^8 = 1$, we have that $X^{24} = 1$.

Therefore $X^{2022} = \left(X^{24}\right)^{84} \cdot X^6 = 1^{84} \cdot X^6 = X^6 = Y^2 = i$,

and $X^{-2022} = \left(X^{2022}\right)^{-1} = i^{-1} = -i$.

Thus $\boxed{X^{2022} + X^{-2022} = 0.}$

Credit goes to Andrew Poe

__Solution #2__: $X^3 + X^{-3} = \sqrt{2} \xrightarrow{\text{square}} X^6 + 2 + X^{-6} = 2 \rightsquigarrow X^6 + X^{-6} = 0$

$\rightsquigarrow X^{12} = 1$.

Then $X^{2022} + X^{-2022} = \left(X^{12}\right)^{168} \cdot X^6 + \left(X^{12}\right)^{-168} \cdot X^{-6} = X^6 + X^{-6} = 0.$