Analyzing Symmetry Groups

1. Let $V$ be the group of symmetries of the rectangle with a side length ratio $2 : 1$. The group $V$ consists of the identity, all rotations, and all reflections of this rectangle.

1.a. Number the vertices of the rectangle to obtain a homomorphism $f : V \to S_4$. List the permutations in the image of this homomorphism.

1.b. Give a presentation of the group $V$ with generators and relations.
2. Let $H$ be the group of all symmetries of the *regular hexagon*. The group $H$ consists of the identity, all rotations, and all reflections of the hexagon.

2.a. List the elements of $H$.

2.b. Give a presentation of the group $H$ with generators and relations.
3. Suppose that $B$ is the group of all rotational symmetries of the rectangular box with side length ratios $2 : 1 : 1$. Number the eight vertices of the box to obtain a homomorphism $\Phi : B \to S_8$.

3.a. List the permutations in the image of this homomorphism.

3.b. Give a presentation of the group $B$ with generators and relations.
The Frobenius Orbit Counting Formula

4. Determine the number of distinct colorings of the eight vertices of a cube, where each vertex is a color from the set \{red, blue, yellow\}. Show your work.

5. Determine the number of distinct colorings of the eight vertices of a rectangular box (of side length ratios $2 : 1 : 1$), where each vertex is a color from the set \{red, blue, yellow\}. Show your work.
6. Determine the number of distinct colorings of the four faces of a tetrahedron, where each face is a color from the set \{orange, purple, black\}. Show your work.

7. Determine the number of six-bead bracelets, where each bead is a color from the set \{red, orange, yellow, green, blue, violet\}. Show your work.
Bonus 1. Determine the number of distinct networks with six nodes. Show your work.
**Bonus 2.** Show that the group of rotational symmetries of the *icosahedron* is isomorphic to the group $A_5$. 