Test 1

ma312: Abstract Algebra I
Northern Michigan University
Fall 2019

- no electronic devices

name: ____________________________________________
1. Consider the operator given by:

\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix} \longrightarrow 
\begin{bmatrix}
    x - y \\
    x + 2y
\end{bmatrix}.
\]

This operator is left-multiplication by some matrix \( A \); find \( A \). Write \( A^{-1} \) and \( A \) as products of elementary matrices. Find \( \det(A) \) and \( \det(A^{-1}) \).
2. Consider the symmetric group $S_5$, with the particular permutation $p = (215)(4215)(325)$.

- write $p$ as a product of disjoint cycles
- state the cycle type of $p$
- write $p$ as a product of **simple** transpositions
- find the permutation matrix $A_p$
- find $\sigma(p)$, the sign of $p$
- write $p^{-1}$ as a product of disjoint cycles
3. Write the permutation \( h = (123)(456) \) as a product of transpositions in three distinct ways. \textit{One of those ways must be a product of simple transpositions.} Find \( \sigma(h) \).
4. Write the elementary matrix $E_{-R_1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ as a product of elementary matrices of the other two types: $E_{R_i+bR_j}$, and $E_{R_i \leftrightarrow R_j}$.

*Bonus.* Show that the elementary matrix $E_{2R_1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ cannot be written as a product of elementary matrices of the other two types.
5. Consider the permutation $\Phi \in S_n$ that is defined by $\Phi(k) = n - k + 1$ for all $k = 1, \ldots, n$. In particular, we have:

$\Phi(1) = n$
$\Phi(2) = n - 1$
$\Phi(3) = n - 2$

\[ \vdots \]
$\Phi(n - 2) = 3$
$\Phi(n - 1) = 2$
$\Phi(n) = 1.$

Find the cycle type of $\Phi$ and the associated permutation matrix $A_\Phi$.

*Bonus.* Find $\sigma(\Phi)$.