A Short Basis for the Variety of Digroups

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Abstract

A digroup is an algebra defined on a set with two associative binary operations. Digroups play a prominent role in an important open problem from the theory of Leibniz algebras. We give a simple basis of independent axioms for the variety of digroups that emphasizes the two core semigroup structures undergirding each algebra in this variety.

One of the most important and challenging open problems in the theory of Leibniz algebras is to find an appropriate generalization of Lie's third theorem. Recall, that Lie's third theorem associates a (local) Lie group to any Lie algebra, real or complex. A particularly vexing aspect of this problem is to find the appropriate analogue of Lie group for Leibniz algebras. Loday [4] found this problem so unyielding that, in a fit of literary reverie, he dubbed these objects "coquecigrues"—imaginary creatures regarded as the embodiment of absolute absurdity. And so the problem has come to be known as *the coquecigrue problem (for Leibniz algebras).* Kinyon has made the best progress to date on this problem by using digroups (definition to follow) to build a partial solution [2]: if a Liebnitz algebra splits over its ideal generated by squares, then it is isomorphic to the tangent algebra of a linear Lie digroup. In particular, if the split Liebnitz algebra is actually a Lie algebra, then the linear Lie digroup coincides with the Lie group from Lie's third theorem. The coquecigrue problem remains open for Leibniz algebras that do not split. For a good overview of the problem see [2].

The notion of digroup first appeared in Loday's work [5]. For other recent work on digroups see [1] and [3]. Kinyon modified Loday's terminology to give a much cleaner definition of digroup (see below), and then used semigroup theory to show that every digroup is a product of a group and a trivial digroup, all in the service of his partial solution to the coquecigrue problem. In this brief note we give an even simpler basis of independent axioms for the variety of digroups.

Since Kinyon's partial solution of the coquecigrue problem, as above, relies on the notion of a linear Lie digroup, for the sake of completeness we offer a definition here, although we won't make use of it in this paper; our focus is on digroups and how they emerge from semigroups. A *Lie digroup* is a smooth manifold with a digroup structure such that the digroup operations are smooth mappings. A *linear Lie digroup* is a Lie digroup recovered from the product of a Lie group and a module in a natural way [2].

Definition 1 (Kinyon) *A* digroup *is a set, G, equipped with two binary operations, $\vdash$ and $\dashv$, with a unary operation, †, and with a nullary operation, 1,* *satisfying each of the following six axioms:*

(G1) $(G, \vdash)$ *and* $(G, \dashv)$ *are both semigroups,*

(G2) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,

(G3) $x \dashv (y \vdash z) = x \dashv y \dashv z$,

(G4) $(x \dashv y) \vdash z = x \vdash y \vdash z$,

(G5) $1 \vdash x = x = x \dashv 1$,

(G6) $x \vdash x^\dagger = 1 = x^\dagger \dashv x$.

We note that clearly $(G, \vdash)$ is a right group and that $(G, \dashv)$ is a left group [2]. Thus, a digroup is an algebra built from a left group structure and a right group structure on the same set, together with a few compatibility conditions. We streamline these conditions by showing that axioms *(G3)* and *(G4)* are superfluous, that *(G2)* can be weakened, and that the axioms in the resulting system are independent.

Theorem 2 *A digroup is a set,* G*, equipped with two binary operations,* $\vdash$ *and* $\dashv$*, with a unary operation,* †*, and with a nullary operation,* 1*, satisfying each of the following four axioms:*

(G1) $(G, \vdash)$ *and* $(G, \dashv)$ *are both semigroups,*

(G2)* $x \vdash (x \dashv z) = (x \vdash x) \dashv z$,

(G5) $1 \vdash x = x = x \dashv 1$,

(G6) $x \vdash x^\dagger = 1 = x^\dagger \dashv x$.

Proof Clearly all digroups satisfy these axioms. Conversely, let G be an algebra satisfying these axioms. We introduce five auxiliary identities to help with the proofs of *(G2), (G3),* and *(G4)*:

(A1) $x^\dagger \vdash x \vdash y = y$,

(A2) $x \vdash x^\dagger \vdash y = y$,

(A3) $(x \vdash y)^\dagger = y^\dagger \vdash x^\dagger$,

(A4) $x \vdash 1 = 1 \dashv x$,

(A5) $(x^\dagger \vdash (x \dashv x)) \dashv y = x^\dagger \vdash (x \dashv (x \vdash y))$.

We show that G must satisfy *(A1)–(A5)*, as well as *(G2), (G3),* and *(G4)*.

(A1): $x^\dagger \vdash x \vdash y \overset{(G5)}{=} x^\dagger \vdash x \vdash 1 \vdash y \overset{(G6)}{=} x^\dagger \vdash x \vdash x^\dagger \vdash x^{\dagger\dagger} \vdash y$ $\overset{(G6)}{=} x^\dagger \vdash 1 \vdash x^{\dagger\dagger} \vdash y \overset{(G5)}{=} x^\dagger \vdash x^{\dagger\dagger} \vdash y \overset{(G6)}{=} 1 \vdash y \overset{(G5)}{=} y$.

(A2) is trivial.

(A3): $(x \vdash y)^\dagger \overset{(A1)}{=} y^\dagger \vdash x^\dagger \vdash x \vdash y \vdash (x \vdash y)^\dagger \overset{(G6)}{=} y^\dagger \vdash x^\dagger \vdash 1$
$\overset{(G6)}{=} y^\dagger \vdash x^\dagger \vdash x \vdash x^\dagger \overset{(A1)}{=} y^\dagger \vdash x^\dagger$.

(A4): $x \vdash 1 \overset{(G6)}{=} x \vdash x^\dagger \vdash x^{\dagger\dagger} \overset{(A2)}{=} x^{\dagger\dagger} \overset{(G5)}{=} x^{\dagger\dagger} \dashv 1 \overset{(G6)}{=} x^{\dagger\dagger} \dashv x^\dagger \dashv x$
$\overset{(G6)}{=} 1 \dashv x$.

(G3): $x \dashv (y \vdash z) \overset{(G5)}{=} x \dashv 1 \dashv (y \vdash z) \overset{(A4)}{=} x \dashv (y \vdash z \vdash 1)$
$\overset{(A1)}{=} x \dashv (y^{\dagger\dagger} \vdash y^\dagger \vdash y \vdash z \vdash 1) \overset{(A1)}{=} x \dashv (y^{\dagger\dagger} \vdash z \vdash 1)$
$\overset{(A4)}{=} x \dashv (y^{\dagger\dagger} \vdash (1 \dashv z)) \overset{(G6)}{=} x \dashv (y^{\dagger\dagger} \vdash (y^{\dagger\dagger} \dashv y^\dagger \dashv z))$
$\overset{(G2^*)}{=} x \dashv (y^{\dagger\dagger} \vdash y^{\dagger\dagger}) \dashv y^\dagger \dashv z \overset{(G2^*)}{=} x \dashv (y^{\dagger\dagger} \vdash (y^{\dagger\dagger} \dashv y^\dagger)) \dashv z$
$\overset{(G6)}{=} x \dashv (y^{\dagger\dagger} \vdash (y^\dagger \vdash y^{\dagger\dagger})) \dashv z \overset{(A1)}{=} x \dashv y^{\dagger\dagger} \dashv z \overset{(G5)}{=} x \dashv (1 \vdash y^{\dagger\dagger}) \dashv z$
$\overset{(G6)}{=} x \dashv (y \vdash y^\dagger \vdash y^{\dagger\dagger}) \dashv z \overset{(G6)}{=} x \dashv (y \vdash 1) \dashv z \overset{(A4)}{=} x \dashv 1 \dashv y \dashv z$
$\overset{(G5)}{=} x \dashv (y \dashv z)$.

(G4): $(x \dashv y) \vdash z \overset{(G5)}{=} (x \dashv y) \vdash 1 \vdash z \overset{(A4)}{=} (1 \dashv x \dashv y) \vdash z$
$\overset{(G3)}{=} (1 \dashv (x \vdash y)) \vdash z \overset{(A4)}{=} x \vdash y \vdash 1 \vdash z \overset{(G5)}{=} x \vdash y \vdash z$.

(A5): $(x^\dagger \vdash (x \dashv x)) \dashv y \overset{(A1)}{=} x^\dagger \vdash x^\dagger \vdash x \vdash x \vdash ((x^\dagger \vdash (x \dashv x)) \dashv y)$
$\overset{(G4)}{=} x^\dagger \vdash x^\dagger \vdash (x \dashv x) \vdash ((x^\dagger \vdash (x \dashv x)) \dashv y)$
$\overset{(G2^*)}{=} x^\dagger \vdash ((x^\dagger \vdash (x \dashv x) \vdash x^\dagger \vdash (x \dashv x)) \dashv y)$
$\overset{(G4)}{=} x^\dagger \vdash ((x^\dagger \vdash x \vdash x \vdash x^\dagger \vdash (x \dashv x)) \dashv y)$
$\overset{(A1)}{=} x^\dagger \vdash ((x \vdash x^\dagger \vdash (x \dashv x)) \dashv y) \overset{(A2)}{=} x^\dagger \vdash (x \dashv (x \vdash y))$.

(G2): $(x \vdash y) \dashv z \overset{(A2)}{=} x \vdash y \vdash y^\dagger \vdash x^\dagger \vdash ((x \vdash y) \dashv z)$
$\overset{(A3)}{=} x \vdash y \vdash (x \vdash y)^\dagger \vdash ((x \vdash y) \dashv z)$
$\overset{(A2)}{=} x \vdash y \vdash (x \vdash y)^\dagger \vdash ((x \vdash y) \dashv ((x \vdash y) \vdash ((x \vdash y)^\dagger \vdash z)))$
$\overset{(A5)}{=} x \vdash y \vdash (((x \vdash y)^\dagger \vdash ((x \vdash y) \dashv (x \vdash y))) \dashv ((x \vdash y)^\dagger \vdash z))$
$\overset{(G3)}{=} x \vdash y \vdash (((x \vdash y)^\dagger \vdash ((x \vdash y) \dashv (x \vdash y))) \dashv (x \vdash y)^\dagger \dashv z)$
$\overset{(A5)}{=} x \vdash y \vdash (((x \vdash y)^\dagger \vdash ((x \vdash y) \dashv (x \vdash y \vdash (x \vdash y)^\dagger))) \dashv z)$
$\overset{(G6)}{=} x \vdash y \vdash (((x \vdash y)^\dagger \vdash ((x \vdash y) \dashv 1)) \dashv z)$
$\overset{(G5)}{=} x \vdash y \vdash (((x \vdash y)^\dagger \vdash (x \vdash y)) \dashv z) \overset{(A3)}{=} x \vdash y \vdash ((y^\dagger \vdash x^\dagger \vdash x \vdash y) \dashv z)$
$\overset{(A1)}{=} x \vdash y \vdash ((y^\dagger \vdash y) \dashv z) \overset{(G5)}{=} x \vdash y \vdash ((y^\dagger \vdash (y \dashv 1)) \dashv z)$
$\overset{(G6)}{=} x \vdash y \vdash ((y^\dagger \vdash (y \dashv (y \vdash y^\dagger))) \dashv z)$
$\overset{(A5)}{=} x \vdash y \vdash ((y^\dagger \vdash (y \dashv y)) \dashv y^\dagger \dashv z) \overset{(A5)}{=} x \vdash y \vdash y^\dagger \vdash (y \dashv (y \vdash y^\dagger \vdash z))$
$\overset{(A2)}{=} x \vdash y \vdash y^\dagger \vdash (y \dashv z) \overset{(A2)}{=} x \vdash (y \dashv z)$.

To establish independence of the axioms we group them as seven axioms and prove the stronger result that these seven are independent:

(G1A) $\vdash$ is associative,

(G1B) $\dashv$ is associative,

(G2)* $x \vdash (x \dashv z) = (x \vdash x) \dashv z$,

(G5A) $1 \vdash x = x$,

(G5B) $x \dashv 1 = x$,

(G6A) $x \vdash x^{\dagger} = 1$,

(G6B) $x^{\dagger} \dashv x = 1$.

Lemma 3 *(G1A), (G1B), (G2*), (G5A), (G5B), (G6A), and (G6B) are independent.*

Proof Let $(G, \dashv, ^{\dagger}, 0)$ be the cyclic group of integers mod 5. Now, let $\vdash$ be a second binary operation on G given by the following Cayley table:

0	1	2	3	4
1	3	4	2	0
2	4	1	0	3
3	2	0	4	1
4	0	3	1	2

$(G, \vdash, \dashv, ^{\dagger}, 0)$ is a model of minimal order satisfying *(G1A), (G1B), (G5A), (G5B), (G6A),* and *(G6B)*, but not *(G2*)*, since $(1 \vdash 1) \dashv 2 = 3 \neq 1 = 1 \vdash (1 \dashv 2)$.

We note that models of size two exist for each of the remaining six cases needed to establish independence. They are trivial to find, and are left to the reader.

References

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