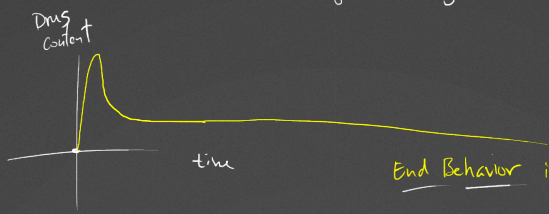
# CHAPTER 4 —

End Behavior: what numbers $y$ $(y = f(x))$ is trending toward as $x$ gets big or small




End Behavior is 0.

To start, let's warm up —

Question: Write down a degree 3 polynomial whose zeros are 1, 3, & 5, w/ leading coef = 5.

( x-intercepts are 1, 3, & 5 )

$y = (x-1)\underbrace{(x-3)(x-5)}$ & multiply out

| | $x$ | $-5$ |
|---|---|---|
| $x$ | $x^2$ | $-5x$ |
| $-3$ | $-3x$ | $15$ |

$(x-1)(x^2 - 8x + 15)$

| | $x^2$ | $-8x$ | $+15$ |
|---|---|---|---|
| $x$ | $x^3$ | $-8x^2$ | $+15x$ |
| $-1$ | $-x^2$ | $8x$ | $-15$ |

$y = x^3 - 9x^2 + 23x - 15$

substitute $x = 1$

$y = 1 - 9 + 23 - 15 = 0$

$x=3$ $\quad y = 27 - 81 + 69 - 15 = 0$

$x = 5$ . . .

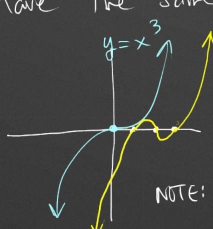
$y = (x^3 - 9x^2 + 23x - 15)\,5 = 5x^3 - 45x^2 + 115 - 75$

In end, this is ignored.

$y = x^3 \boxed{- 9x^2 + 23x - 15}$ The end behavior is entirely controlled by the leading term.

$y = x^3$ &

have the same end behavior.




as $x$ goes right, $y$ goes up

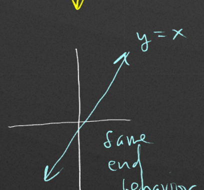
as $x \to \infty$, $y \to \infty$

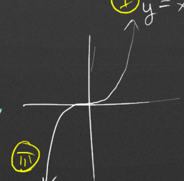
$x \to -\infty$, $y \to -\infty$

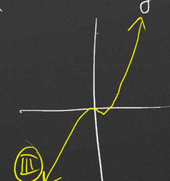
as $x$ goes left, $y$ goes down

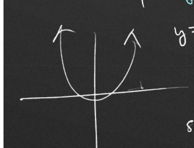
NOTE: leading coeff is positive! ← pay attn to this

(I) $y = x^5$ (I) $y = x^7 - x^4 + 1$




as $x \to \infty$ $\quad y \to \infty$

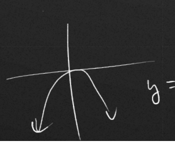
$x \to -\infty$ $\quad y \to \infty$

move up on both ends

same for

$y = (x-1)^4$

(all even degree)




as $x \to \infty$, $y \to -\infty$

$x \to -\infty$, $y \to -\infty$.

For Rational Functions, again only the leading terms matter

$$f(x) = \frac{x^2+1}{2x^2-7}$$

when $x \to \infty$ (means x is ****-ing large)

$x+1$

if $x = 100{,}000{,}000$ you're considering

$100{,}000{,}000$ vs. $100{,}000{,}001$

$$f(x) \approx \frac{x^2}{2x^2} = \frac{1}{2}$$

algebra

So here

as $x \longrightarrow \infty$, $f(x) \longrightarrow \frac{1}{2}$

(horizontal asymptote at $y = \frac{1}{2}$)

Ex: Determine the end behavior of when $x \to \infty$

$$f(x) = \frac{x^3 + 1000x^2 + 231x + 57}{10x^3 + 17} \approx \frac{x^3}{10x^3} = \boxed{\frac{1}{10}}$$

NO X to sub in

as $x \longrightarrow \infty$ $\quad f(x) \longrightarrow \frac{1}{10}$

as $x \longrightarrow -\infty$ $\quad f(x) \longrightarrow \frac{1}{10}$

only leading terms matter

$$f(x) = \frac{x^3 + 1000x^2 + 231x + 57}{10x^2 + 17};$$

when $x \to \infty$

$$\approx \frac{x^3}{10x^2} = \frac{x}{10}$$

algebra step

So now $f(x) \approx \frac{x}{10}$

as $x \longrightarrow \infty$ $\quad f(x) \longrightarrow \infty$

as $x \longrightarrow -\infty$ $\quad f(x) \longrightarrow -\infty$

$$f(x) = \frac{x^3 + 1000x^2 + 231x + 57}{20x^3 + 10x^2 + x + 1};$$

now notice leading term in num $= x^3$

leading term in den $= 20x^3$

$$\frac{x^3}{20x^3} = \frac{1}{20}$$

$$f(x) = \frac{x^3 + 1000x^2 + 231x + 57}{10x^4 + 17}$$

leading terms : $\frac{x^3}{10x^4} \approx f(x)$

algebra

$$\frac{1}{10x} \xrightarrow{\text{goes to}} 0$$

as $x \to \infty$

1.0e-8

= .00000001

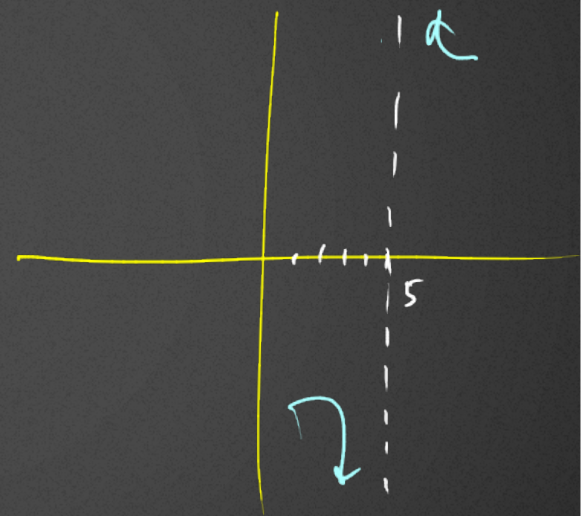
very close to 0

(divide by bigger & bigger #'s your quotient gets smaller)

End Behavior describes the horizontal asymptotes.

# VERTICAL ASYMPTOTE

$$f(x) = \frac{x^2 - 1}{x - 5}$$




To find: set den $= 0$

$$x - 5 = 0$$

$$\boxed{x = 5}$$

eqn of the vertical asymptote

$$f(4.9) = \frac{(4.9)^2 - 1}{4.9 - 5} \approx \frac{25 - 1}{-.1} = \frac{24}{-.1} = -240$$

$$f(5.1) = \frac{(5.1)^2 - 1}{5.1 - 5} = \frac{25 - 1}{.1} = 240$$

# CHAPTER 4 -

End Behavior / Asymptotes
what are the
y-values doing (trending toward)
as x gets big ($x \to \infty$)
& gets small ($x \to -\infty$)

For starters, let's warm-up —

1. Write down a degree 3 polynomial whose "zeros" are 1, 4 & 7. (leading coeff = 1)
(Means the graph will cross x-axis at 1, 4 & 7)
(Means x-intercepts are 1, 4, 7)

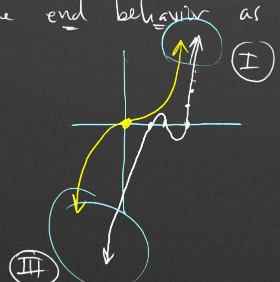
start
$$y = (x-1)\underbrace{(x-4)(x-7)}$$
then multiply this out.
$$= (x-1)(x^2 - 11x + 28)$$
$$= x^3 - 11x^2 + 28x - x^2 + 11x - 28$$
$$= \boxed{x^3} - 12x^2 + 39x - 28$$
leading term !! (highest exponent)

plug in $x = 1$ $= 1 - 12 + 39 - 28 = 0$

same for $x = 4$, & $x = 7$.

Cool Fact: Leading term governs / controls / determines the end behavior.

I mean: $y = x^3 - 12x^2 + 39x - 28$ has the same end behavior as $y = x^3$.




So we say $y = x^3 - 12x^2 + 39x - 28 \xrightarrow{\text{y goes up}} \infty$
as $x \to \infty$ (x goes right)

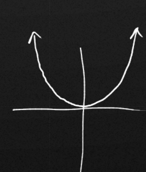
$y \to -\infty$ (y goes down)
as $x \to -\infty$ (x goes left)

for (positive leading term)

This is the same type of end behavior for
$y = x$, $y = x^5$, $y = x^7$, $y = x^9$
odd degree polynomials

On the other hand —

$y = x^2$, $y = x^4$, $y = x^6$, ...



Polynomial End Behavior -

Rational Functions:

$$f(x) = \frac{x^2+1}{x-2}$$

End Behavior

Imagine $x \to \infty$. This $x$ is so large that only the leading terms matter.

(100,000 is basically the same as 100,001)

So as $x \to \infty$ $f(x)$ behaves like $\frac{x^2}{x} = x$

$f(x) \to \infty$

as $x \to -\infty$

$f(x) \to -\infty$.

ignore all but the leading terms.

Ex $f(x) = \frac{1-x^2}{x+100} \approx \frac{-x^2}{x} = -x$

as $x \to \infty$, $f(x) \to$ $\underline{-\infty}$

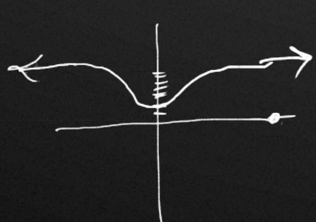
as $x \to -\infty$, $f(x) \to$ $\underline{\infty}$

Ex. $f(x) = \frac{10x^2+5}{x^2+50}$

as $x \to \infty$, $f(x) \to$ $\underline{\frac{10x^2}{x^2} = 10}$

as $x \to -\infty$, $f(x) \to$ $\underline{\frac{10x^2}{x^2} = 10}$

no space to put in $x$. so end behavior is $\to 10$.



domain $= \mathbb{R}$ $x^2+50 \neq 0$

Ex.

$$\qquad\quad 3 \qquad 56 + 4000 - 200 + 1$$

$$f(x) = \frac{7x^3 + 1000x^2 - 100x + 1}{x^3 - 8}$$

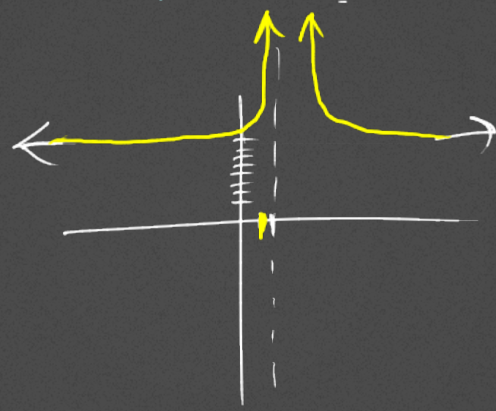
<u>domain</u>

$\mathbb{R} - \{2\}$

$x^3 - 8 = 0$

$\Rightarrow x = 2.$

(we know the graph isn't there (defined) at $x = 2$)



$f(2) = \frac{3700}{0}$

$f(2.1)$

$f(1.9)$

Is this a hole or a vertical asymptote?


(usually a vert. asymptote.
$\therefore$ only get a hole when the denominator cancels with a factor of the numerator
(eg. $f(2) = \frac{0}{0}$)

Ex. $f(x) = \dfrac{3x^2 - 1}{x+1}$

**End Behavior**: (Includes finding horizontal asymptotes)

as $x \to \infty$, $f(x) \approx \dfrac{3x^2}{x} = 3x$.

so as $x \to \infty$, $f(x) \to \infty$

$x \to -\infty$, $f(x) \to -\infty$

**Vertical Asy**

domain: $\mathbb{R} - \{-1\}$.

vertical asy @ $\underline{x = -1}$ because $\dfrac{3(-1.1) - 1}{-1.1 + 1} \neq 0$