$a > 0$ $\quad a^x = -1 \Rightarrow$ no! $\Rightarrow$ domain $\log_a(x)$ is $\underline{\mathbb{R}^{>0}}$.

$$a^x = b \iff \log_a b = x$$

solve: $a^x = 0$ $\Rightarrow$ NOT POSSIBLE! $\Rightarrow \log_a 0 = x$ has no sols.

$$\log_a M^N = \underline{\log_a (a^x)^N} = \log_a a^{x \cdot N} = y$$

$$\log_a M = x \Rightarrow a^x = M$$

$$\Rightarrow a^y = a^{x \cdot N}$$

$$\Rightarrow y = x \cdot N$$

or $y = N \cdot \log_a M$

$$\log_a (M \cdot N) = \underline{\log_a (a^x \cdot a^y)} = \log_a a^{x+y} = \boxed{w}$$

$$\Rightarrow a^w = a^{x+y}$$

$\log_a M = x$, $\log_a N = y$

$\Rightarrow a^x = M$ & $a^y = N$ $\quad \Rightarrow \boxed{w} = x + y = \log_a M + \log_a N$

Ex

$$\log(x+1) + \log(x-1) = 0$$

$$\Rightarrow \log((x^2 - 1)) = 0$$

$$\Rightarrow x^2 - 1 = 1 \Rightarrow x^2 = 2, \quad x = \pm\sqrt{2}$$

But domain log is $\mathbb{R}^{>0}$

$\Rightarrow x = \sqrt{2}$

$A = Pe^{rt}$. $\quad P = 1000$, $r = 5\%$. How long till double?

$A = 2000$. $\quad 2000 = 1000\, e^{.05t}$

$$2 = e^{.05t} \iff \ln(2) = .05t$$

$$\Rightarrow \frac{\ln(2)}{.05} = t$$

CHAPTER 5

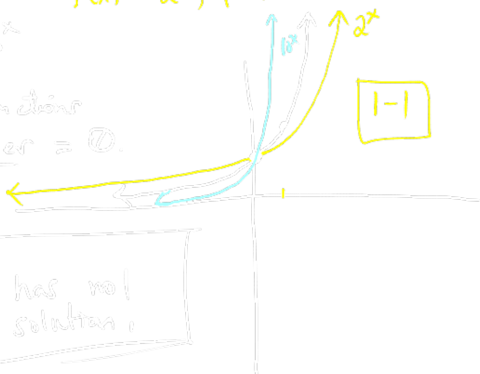
up to now ... polynomial, rational functions.

today: exponential & logarithmic functions

ex: $f(x) = 2^x$, $f(0) = 2^0 = 1$ (2 is the "base")

ex: $f(x) = 3^x$

These functions are never $= 0$.



$\boxed{1-1}$

$\boxed{2^x = 0 \text{ has no solution.}}$

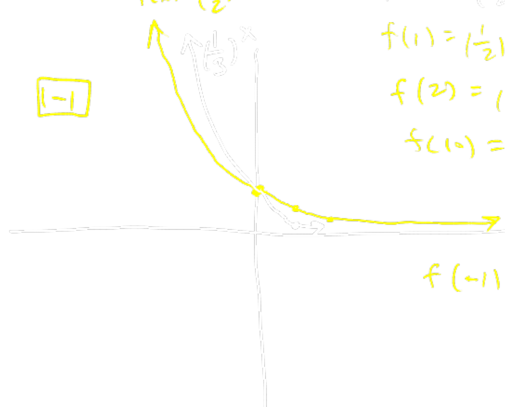
$3^1 = 3$
$3^0 = 1$
$2^0 = 1$
$2^1 = 2 = 2$
$2^2 = 2 \cdot 2 = 4$
$2^3 = 2 \cdot 2 \cdot 2 = 8$
$2^{-1} = \frac{1}{2}$
$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
$3^{-1} = \frac{1}{3}$

$f(x) = 1^x$ is ⟷

$f(x) = \left(\frac{1}{3}\right)^x$

$f(1) = \frac{1}{3}$

$f(x) = \left(\frac{1}{2}\right)^x$



$\boxed{1-1}$

$f(x) = \left(\frac{1}{2}\right)^x$

$f(0) = \left(\frac{1}{2}\right)^0 = 1$

$f(1) = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

$f(2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$f(10) = \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}} = 1024$

$f(-1) = \left(\frac{1}{2}\right)^{-1} = \left(\frac{2}{1}\right)^1 = 2$

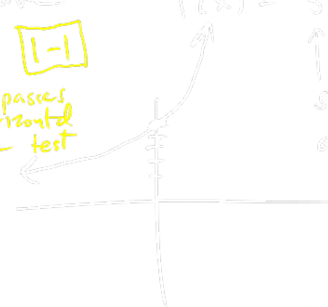
Also, we have $f(x) = 5 \cdot 2^x$ $\boxed{\neq 10^x}$

$f(0) = 5$

$f(1) = 10$

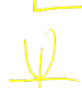
$\boxed{1-1}$

passes horizontal line test



shifts\scales the graph of 2^x up/down

All exponential functions are $\boxed{1-1}$.

(as opposed to $f(x) = x^2$ )

THEY HAVE INVERSES!

The logarithmic function is defined as follows —

$$\boxed{\log_a x = b \quad \underset{\text{when}}{\overset{\text{precisely}}{\Longleftrightarrow}} \quad a^b = x}$$

"log base a of x equals b"

This defines $\log_a x$ as the inverse of a^x.

This allows us to solve exponential eqn's.

Ex Solve

$$2^x = 100.$$

Apply $\log_2(x)$ to both sides.

$$\underbrace{\log_2(2^x)}_{= x} = \log_2(100)$$

$$= 6.64$$

because this is $f^{-1}(f(x)) = x$

Check

$$2^{6.64} = 99.7$$

✓

① set equal to y ② apply def.

Ex.

$\log_{10}(10) = y$ MEANS $10^y = 10 \Rightarrow y = 1$

= ① (from definition above)

$\log_{10}(100) = 2$

$\log_{10}(1000) = 3$

$\ln(x) = \log_e(x)$

Practice: Solve.

$\ln(x)$ is the

① $\underbrace{\ln(e^x)}_{x} = \ln(75) \approx 4.3$ (Hint: inverse of e^x

② $\dfrac{10^x + 1}{7} = 100$ } $10^x + 1 = 700$

$\underbrace{\log_{10}(10^x)}_{= x} = \log_{10}(699)$

$= x \approx 2.8$

check

$\dfrac{10^{2.8}+1}{7}$

③ $\left(\log_{10}(x+1)\right) = (3)$

10 10

equiv. to

$10^3 = x+1$

$999 = x$

$x + 1 = 10^3$

$x = 999$

Think: what's happening to x?

A: inside $\log_{10}$.

Two important properties of log(x)

$$\boxed{\log_a(X^Y)} = \log_a((a^W)^Y) = \log_a(a^{WY}) = Z$$

means

To figure this out examine $a^Z = a^{WY}$

$$\boxed{\log_a X = W} \quad \Rightarrow Z = WY \quad \boxed{Z = Y \cdot \log_a X}$$

means $\boxed{a^W = X}$ ... go back & substitute

So ______

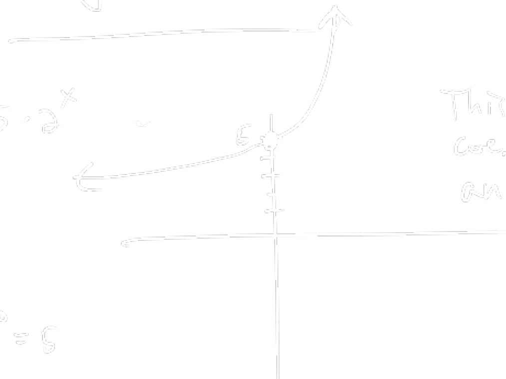
$$\boxed{\log(X^Y) = Y \cdot \log X}$$

General Exponential Equations. $a > 1$

$$y = C \cdot a^x$$

Ex. $y = 5 \cdot 2^x$

graph



This leading coef is often an initial condition

@ $x = 0$ $y = 5 \cdot 2^0 = 5$

@ $x = 1$ $y = 5 \cdot 2^1 = 10$

$$\boxed{\log_a(x) = b \quad \text{means} \quad a^b = x}$$

This just means that $\log_a x$ & a^x are inverse functions

- Also means that $\log_a x$ is an exponent.

So $a^{\log_a(x)} = x$

i.e., $a^{\log_a(\text{Donald Trump})} = \text{Donald Trump}$

Ex. $2^{\log_2 3}$ $\underline{\quad 3 \quad}$

$10^{\log 7}$ $\underline{\quad 7 \quad}$

$e^{\ln(15)}$ $\underline{\quad 15 \quad}$

what is

$$\log_a(x^y) \qquad ?$$

To start look at $\boxed{\log_a x = w.}$

I don't know what this is, but I name it w.

this means

$$a^w = x$$

let it = z

So $\log_a((a^w)^y) = \log_a(a^{wy}) = z = wy$

$$\log_a x^y = y \cdot \log_a x$$

Point: Exponents come down in front!

$$\boxed{\log(x^y) = y \cdot \log x}$$

So:

$$\log_{10}(2^x) = 5$$

$$x \cdot \log_{10}(2) = 5$$

$$x = 5/\log_{10}(2)$$

Practice:

$$\ln e^{-4} = \underline{-4 \cdot \ln(e) = -4}$$

$$\ln e^{4} = \underline{4 \cdot \ln(e)} = 4$$

$$\ln\left(\frac{1}{e}\right) = \underline{\ln\left(e^{-1}\right)} = -1 \cdot \ln(e)$$

$$= -1$$

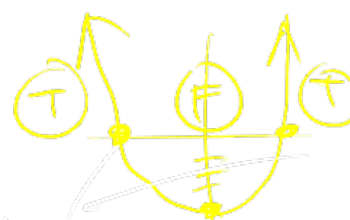
Recall $\ln(x)$ & e^x are inverse functions
so the range of e^x is the domain of $\ln(x)$.

range$(e^x) = (0, \infty) =$ domain$(\ln(x))$
same for all bases

Ex. Domain $f(x) = \log(x^2 - 4)$

Think: Inside parenthesis must be > 0.

$$x^2 - 4 > 0$$



$(-\infty, -2) \cup (2, \infty)$

$x^2 = 4$
$x = \pm 2$

Ex $\log\left(\underbrace{\log 1000^{10000}}\right)$

understood base $= 10$

$\log\left(\overset{A}{10000}\ \overset{B}{\log 1000}\right)$

$\boxed{\log_{10} 10000} + \log_{10}\left(\underbrace{\log_{10} 1000}_{3}\right)$

$\parallel$

$\boxed{4 + \log_{10} 3}$

Hint: $\log A^B = B \cdot \log A$ ⊛

⊛ $\log_{10}(AB) = \log_{10}(A) + \log_{10}(B)$

why?

raise all sides to powers of 10

$\cancel{10^{\log_{10}}}AB = 10^{(\log_{10} A + \log_{10} B)}$

$= \cancel{10^{\log}}A \cdot \cancel{10^{\log}}B$

$AB = A \cdot B$

$$\underbrace{\log_2(x) = y}_{\text{means}}$$

To find y without $\log_2$ button...

$$2^y = x$$

$$\ln(2^y) = \ln(x)$$

$$y \cdot \ln(2) = \ln(x)$$

$$\boxed{y = \frac{\ln(x)}{\ln(2)}}$$

Properties of $\log_a(x)$.

- $\log_a(x) = b$ means $a^b = x$
- $\log_a(a^x) = x \cdot \underbrace{\boxed{\log_a(a)}}_{=1} = x$
- ☆ $\log(xy) = \log x + \log y$
- $\log\left(\frac{x}{y}\right) = \log x - \log y$

$$\left(\log_a(x) + \log_a(y)\right) = w$$ (give LHS a name w)

$$a \qquad\qquad a$$

$$\underbrace{a^{\log_a(x)}} \cdot a^{\log_a(y)} = a^w$$

$$\log_a(x \cdot y) = \log_a(a^w) = w$$

$$\log_a(xy) = w$$

$\log x + \log y = \log(x \cdot y)$

- $\boxed{a^{\log_a(x)} = x}$ why?

B/C

$a^{\log_a(x)} = w$ means $\log_a w = \log_a x$

so $w = x$!!

log is 1-1

use some properties ——

solve: $\left(\log_{10}(x^2-1)\right) = 0$ (understood 10 is the base).

$10^{\left(\log_{10}(x^2-1)\right)} = 10^{0}$

$x^2-1 = 10^0 = 1$

$x^2 = 2 \Rightarrow x = \pm\sqrt{2}$.

check: $\log\left((-\sqrt{2})^2-1\right) = \log(2-1) = \log(1) = 0$

Rewrite as one logarithm ——

$\underbrace{4\cdot\log x}_{\shortparallel} - 5\log(x^2+1) + 4\log(x-1)$

$\log x^4 - \log(x^2+1)^5 + \log(x-1)^4$

$\log\left(\frac{x^4}{(x^2+1)^5}\right) + \log(x-1)^4$

$= \log\left(\frac{x^4\cdot(x-1)^4}{(x^2+1)^5}\right)$

Practice   Simplify

(a) $\log_7 7^{(-2x-10)} = -2x-10$

$$2^4 = 2^{4\cdot 9\cdot\frac{1}{9}} = 2^{9\cdot 4/9}$$

(b) $4^{\log_4(5+8x)} = 5+8x$

$512 = 2^9$ log form

(c) $\log_{512} 16^k = k\cdot\log_{512}16 = k\cdot\boxed{\log_{2^9} 2^4}$

$(2^9)^p = 2^4$

$2^{9p} = 2^4$

$9p = 4$

$p = 4/9$

$= k\cdot\cancel{\log_{2^9}}\,\cancel{(2^9)}^{4/9} = \boxed{4/9\,k}$

(d) $8^{5\log_8 9 - 9\log_8 5}$

$8^{\log_8 9^5 - \log_8 5^9} = \cancel{8^{\log_8}}\left(\frac{9^5}{5^9}\right) = \frac{9^5}{5^9}$

(e) $\log_{32} 16^k = p$ rewrite in exp. form

what must p be?   means $32^p = 16^k$

$(2^5)^p = (2^4)^k$

$5p = 4k$ $2^{5p} = 2^{4k}$

$p = 4/5\,k$ now since bases are equal, exponents must be equal

(c) $\log_{512} 16^k = p$ means $512^p = 16^k$

$(2^9)^p = (2^4)^k$

this means $2^{9p} = 2^{4k}$

$9p = 4k$

$p = 4/9\,k$

Rewrite as 3 log functions

$$\log\left(\frac{x^{17}\cdot y^{8}}{z^{9}}\right) = \underbrace{\log\left(x^{17}\cdot y^{8}\right)} - \log\left(z^{9}\right)$$

$$= \log(x^{17}) + \log(y^{8}) - \log(z^{9})$$

$$= 17\cdot\log x + 8\log y - 9\log z$$

$$= A\log(x) + B\log(y) + C\log(z)$$

$A = 17,\ B = 8,\ C = -9.$

Ch.5 - exp & log functions . WeBWork is up.

$2^{10} = 1024$
$2^9 = 512$
$2^8 = 256$

Ex

$$\log_2 32 + \log_2 8 = \log_2 256$$
$$5 + 3 = 8$$

$$\log A + \log B = \log AB$$

Ex.

$$2^{(\log_2 64)} = 2^6 = 64$$

$$3^{(\log_3 81)} = 3^4 = 81$$

Practice: simplify

(a) $\log_7 7^{-2x-10} = p$

this means $7^p = 7^{-2x-10}$

$p = -2x-10$

Hint:
$\log AB = \log A + \log B$
$\log\left(\frac{A}{B}\right) = \log A - \log B$

(b) $4^{\log_4(5+8x)} = 5+8x$

(c) $\log_{64} 16^k = p \Rightarrow 64^p = 16^k$

$(2^6)^p = (2^4)^k$ bases same

$2^{6p} = 2^{4k} \Rightarrow 6p = 4k$

$p = 4/6\, k$

(d) $8^{5\log_8 9 - 9\log_8 5}$

(a) $\log_2 2 = 1$

$\log_2 2^4 = 4$

$\log_7 7^{4} = 4$

$\log_7 7^{(-2x-10)} = -2x-10$

(b) $4^{\log_4 16} = 16$, $\log_4 16 = 2$

(d) $8^{5\log_8 9 - 9\log_8 5}$

$\log(A) - \log(B) = \log\left(\frac{A}{B}\right)$

$$= 8^{\log_8 9^5 - \log_8 5^9}$$

$$= \cancel{8^{\log_8}}\frac{9^5}{5^9} = \frac{9^5}{5^9}$$

$\log(\text{I'm}) - \log(\text{You}) = \log\left(\frac{\text{I'm}}{\text{You}}\right)$

Ex: Rewrite into a single log.

$$4\log x - 5\log(x^2+1) + 4\log(x-1)$$

$$\underbrace{\log x^4 - \log(x^2+1)^5}_{\log\left(\frac{x^4}{(x^2+1)^5}\right)} + \log(x-1)^4$$

$$\log\left(\frac{x^4}{(x^2+1)^5}\right) + \log(x-1)^4$$

$$\log\left(\frac{x^4}{(x^2+1)^5}\cdot(x-1)^4\right) = \log\left(\frac{x^4(x-1)^4}{(x^2+1)^5}\right)$$

True or False

(a) $$\frac{\ln(x)}{\ln(y)} = \ln(x) - \ln(y)$$

$\ln(e) = p$

$\Leftrightarrow$

$e^p = e$

$\boxed{p = 1}$

if $y = e$ then this would imply

$$\frac{\ln(x)}{\ln(e)} = \ln(x) - \ln(e)$$

NO

$$\frac{\ln(x)}{1} = \ln(x) - 1 \qquad \times$$

$\cdot\ \ln(a^b) = b \cdot \ln a$ always true.

$$\ln(x-y) = \ln(x) - \ln(y) \ ?$$

if $x = y$ false

(L) $\ln(x-x) = \ln(0)$ DNE.

(R) $\ln(x) - \ln(x) = 0$

(c) $$(\ln a)^b \overset{?}{=} b \ln a = \ln a^b \qquad \text{False}$$

$a = e$, $b = 2$ } $\underbrace{(\ln(e))^2}_{(1)^2 = 1} \overset{?}{\cancel{=}} \ln(e^2) = 2 \cdot \ln e = 2$

Break into 3 logs

$$\log\left(\frac{x^{17}y^{8}}{z^{9}}\right) = \log\left(\overset{\text{num}}{x^{17}y^{8}}\right) - \log\left(\overset{\text{den}}{z^{9}}\right)$$

$$\log x^{17} + \log y^{8} - \log z^{9}$$

$$A\log x + B\log y + C\log z$$

$$= 17\log x + 8\log y - 9\log z$$

$A = 17,\ B = 8,\ C = -9$

5.2

$\log_2 1 = 0$

$\log_2 2 = 1$

$\log_2 4 = 2$

$\log_2 64 = 6$

$\log_2 \left(\frac{1}{2}\right) = -1$

$\log_2 \left(\frac{1}{64}\right) = -6$

$\log_3 3 = 1$

$\log_3 27 = 3$

$\log_3 81 = 4$

$3^{p} = \frac{1}{3} = 3^{-1}$

$\log_3 \frac{1}{3} = p = -1$

$\log_5 25 = 2$

$\log_{10} 100 = 2$

$\boxed{\log_a x = b \text{ means } a^b = x}$

$\log .1 = \log_{10} \left(\frac{1}{10}\right) = -1$

$\log_{10} 1000 = 3$

---

logs are often used to make it easier to deal with large numbers.

$\log a^x = x \log a$

Ex. googol $= 10^{100}$.

$\underbrace{10000 \ldots 0}_{\leftarrow 100 \rightarrow}$

$$\log(\log(\text{googol})) = \log(\log(10^{100}))$$

$$= \log(100 \cdot \underbrace{\log(10)})$$

$$= \log(100 \cdot 1) = \log(100)$$

$$= 2$$

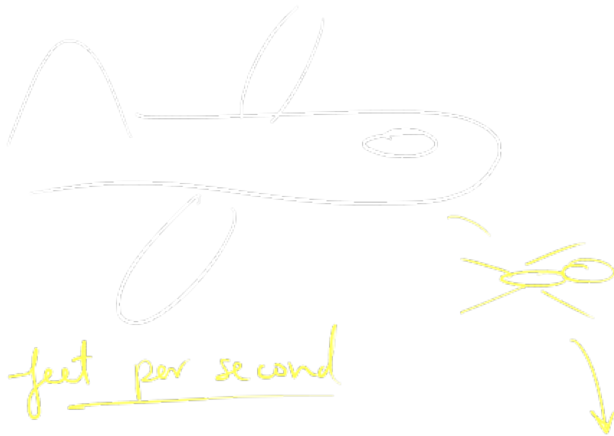
$$\text{googolplex} = 10^{\text{googol}} = \underbrace{1000 \ldots 0}_{\leftarrow \text{googol} \rightarrow}$$

$$\log(\log(\log(\text{googolplex}))) = \log(\log(\log(10^{\text{googol}})))$$

$$= \log(\log(\text{googol} \cdot \underbrace{\log 10}_{=1}))$$

$$= 2$$

$$\boxed{10^{10^{10^2}} = \text{googolplex}}$$



feet per second

— velocity of a skydiver —

$$v(t) = 80\left(1 - e^{-.02t}\right)$$

Q1: Initial Velocity?

$$v(0) = 80\left(1 - \boxed{e^{-.02(0)}}\right)$$

$$= 80(1-1)$$

$$= 0!$$

Q2: How fast we're falling 10 secs after jump?

$$t=10 \Rightarrow v(10) = 80\left(1 - e^{-.2}\right) = 14.5 \text{ feet/sec}$$

Q3: what's the terminal velocity! A: $v(t) \longrightarrow$ Terminal Velocity as $t \longrightarrow \infty$

small as $t \to \infty$

$$v(t) = 80\left(1 - \frac{1}{e^{.02t}}\right)$$

imagine $v(t)$ for $t = 100, 1000, 10000, 100{,}000$

as $t \to \infty$ $v(t) \to 80(1-0) = 80$

Fastest Ever: $1357 \frac{km}{h} \cdot \frac{1000\ m}{1\ km} \cdot \frac{3\ ft}{m} \cdot \frac{mi}{5280}$

≈ 771 m/h

How loud are our headphones?

Google:

105 – 120 dB (decibels)
when vol is all way up.

$$dB = 10 \cdot \log\left(\frac{I}{I_0}\right)$$

I = physical intensity of sound (headphones)
I_0 = phys. intensity of a barely audible sound.

Q: How many decibels is a barely audible sound?
A: set $I = I_0$

$$dB = 10 \cdot \log\left(\frac{I_0}{I_0}\right) =$$

$$= 10 \cdot \log(1) = 0$$

Q: How much more intense is your headphone volume (120 dB) than a barely audible sound.

set $I = k \cdot I_0$

$$120 = 10 \cdot \log\left(\frac{I}{I_0}\right) = 10 \log\left(\frac{kI_0}{I_0}\right)$$

$$120 = 10 \log(k)$$

$60 = 10 \log k$

$6 = \log k$

$10^6 = k$

$12 = \log(k)$

$10 \quad\quad 10$

$\Rightarrow 10^{12} = k$

Trillion times more powerful than a barely audible sound.

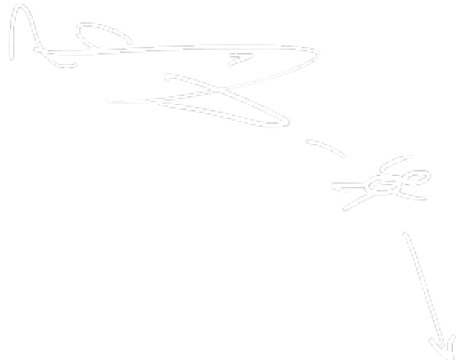
whisper : 30 dB

hmmm : 40 dB

normal speech : 60 dB

Also: Q: how much more powerful are your headphones than normal speech?

$60 =$



velocity of the sky diver:

$$v(t) = 80\left(1 - e^{-.2t}\right)$$

feet/sec.

---

1. what's the initial velocity $(t=0)$

$$v(0) = 80\left(1 - e^{0}\right)$$
$$= 80(1-1) = 0$$

2. what's the terminal velocity

(as time gets large $t \to \infty$ $v(t) \longrightarrow$ terminal velocity)

$$v(t) = 80\left(1 - \overbrace{e^{-.2t}}^{\text{dies as } t \text{ grows.}}\right)$$

$$v(1000) = 80\left(1 - \frac{1}{e^{.2(1000)}}\right) = 80\left(1 - \frac{1}{e^{200}}\right)$$

$$= 80\left(1 - \frac{1}{\text{HUGE \#}}\right)$$

$$\approx 80(1 - 0) = 80$$

World Population:

One argument: pop is leveling off.

$$P(t) = \frac{73.2}{6.1 + 5.9e^{-.02t}}$$ in billions

t is years since 2000

$$= \left(\frac{73.2}{6.1 + \frac{5.9}{e^{.02t}}} \right)$$

1. What was pop in 2000, $P(0) = \frac{73.2}{6.1 + 5.9e^{0}}$

$= 6.1$ billion

2. pop in 2015, $P(15) = \frac{73.2}{6.1 + 5.9e^{-.02(15)}} = 6.9$

3. long term projected pop. (terminal pop.)

$P(t) \longrightarrow$ what as $t \to \infty$

$$\longrightarrow \frac{73.2}{6.1 + \frac{5.9}{\text{HUGE}}} \approx \frac{73.2}{6.1 + 0} = \underline{12 \text{ billion}}$$

decibel level in your headphones

$dB \approx 105 - 120$ decibels

when you turn volume all way up ..

$$dB = 10 \cdot \log\left(\frac{I}{I_0}\right)$$

I = intensity of headphone sound

I_0 = intensity of a barely audible sound.

1. what decibel level does a "barely audible sound" have. set $I = I_0$

$$dB = 10 \cdot \log\left(\frac{I_0}{I_0}\right) = 10 \log(1) = 10 \cdot 0 = 0$$

2. How much more intense is your headphone sound level than a barely audible sound.

set $I = kI_0$ find k given $dB = 120$.

$$120 = 10 \cdot \log\left(\frac{I}{I_0}\right)$$

$$= 10 \cdot \log\left(\frac{kI_0}{I_0}\right) = 10 \cdot \log(k)$$

$$120 = 10 \cdot \log(k)$$

$$12 = \log(k)$$

so $10^{12} = k$

10^{12} times more intense than a barely audible sound.

$$\log_2 64 = p \Rightarrow 2^p = 64$$