Final Exam Guide :: Math 115 **Name** Solutions

1. Evalue the function below at $f(-5), f(0), f(1), f(2), f(5)$

$-15 \quad 1 \quad 2 \quad 3 \quad 9$

$$f(x) = \begin{cases} 3x & \text{if } x < 0 \\ x+1 & \text{if } 0 \leq x \leq 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$$

2. Simplify

$$(x+y)^2 - x^2 - y^2 = 2xy$$

3. Factor

$$yx^2 + y + 2y^2(x^2+1)$$

$$y(x^2+1) + 2y^2(x^2+1)$$

$$(y+2y^2)(x^2+1) = \boxed{y(1+2y)(x^2+1)}$$

4. Factor

$$x^3 - 1 \quad \text{and} \quad x^3 + 1$$

$x^3 - 1 = (x-1)(x^2+x+1)$

$x^3 + 1 = \boxed{(x+1)(x^2-x-1)}$

You know -1 is a root, so $x+1$ is a factor.

$$\begin{array}{r} x^2 - x - 1 \\ x+1 \overline{) x^3 \qquad +1} \\ \underline{x^3 + x^2} \\ -x^2 \\ \underline{-x^2 - x} \\ -x \\ \underline{-x - 1} \\ 0 \end{array}$$

5. Factor

know $a-b$ is a factor

$$\begin{array}{r} a^2 + ab + b^2 \\ a-b \overline{) a^3 \qquad - b^3} \\ \underline{-a^3 - a^2b} \\ a^2b \\ \underline{-(a^2b - ab^2)} \\ ab^2 \\ \underline{-(ab^2 - b^3)} \\ 0 \end{array}$$

$$a^3 - b^3$$

$$= \boxed{(a-b)(a^2+ab+b^2)}$$

6. Simplify

$$\underbrace{8x^3 - 27}_{} - \underbrace{(2x-3)(4x^2+6x+9)}_{\text{same as this}} = 0$$

$= (2x)^3 - 3^3$

$= (2x-3)(4x^2+6x+9)$ so

7. Factor completely

$$(x+1)^{1/2}y + 2y^2(x+1)^{3/2}$$

factor out the common terms with the smallest exponents.

$$y(x+1)^{1/2}\left[1 + 2y(x+1)^{\frac{3}{2}-\frac{1}{2}}\right]$$

when you factor you subtract exponents

$$= \boxed{y(x+1)^{\frac{1}{2}}\left(1+2y(x+1)\right)}$$

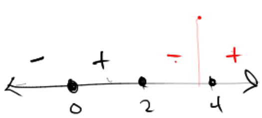
8. Solve the inequality $f(x) < 0$, and sketch a graph of $f(x)$ if

$$f(x) = \frac{x}{x^2 - 6x + 8}$$

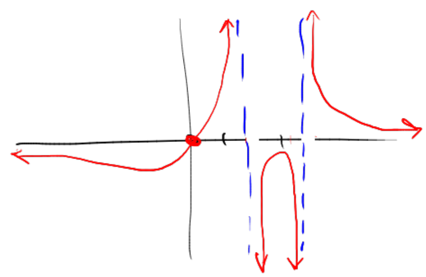
solution: $(-\infty, 0) \cup (2, 4)$

Zeros: $x = 0$

break points: $x^2 - 6x + 8 = 0$
$x = 4, 2$



x	$f(x)$
-100	$\approx (-100)/(-100)^2 = -1/100 < 0$ (close to zero)
1	$1/3 > 0$
3	$3/(9-18+8) = 3/-1 < 0$
100	$\approx 100/100^2 = 1/100 > 0$ (close to zero)



9. Find the equation of the line which passes through $(1, 5)$ and is parallel to $y = 3x + 4$.

$m = 3$

$y - 5 = 3(x - 1)$

$\boxed{y = 3x + 2}$

10. Find the equation of the perpendicular bisector of the line segment AB where $A = (10, -5)$ and $B = (-8, 11)$.

$m_{AB} = \frac{-5 - 11}{10 - (-8)} = \frac{-16}{18} = \frac{-8}{9} \Rightarrow m_{\perp} = \frac{9}{8}$

$\text{midpoint}_{AB} = \left(\frac{10 - 8}{2}, \frac{-5 + 11}{2}\right) = (1, 3)$

$y - 3 = \frac{9}{8}(x - 1)$ or $y = \frac{9}{8}x - \frac{9}{8} + \frac{24}{8} \Rightarrow \boxed{y = \frac{9}{8}x + \frac{15}{8}}$

11. The depth in (inches) of snow during a major UP mid-winter storm is given by the function $d(t) = 3t + 22$ where t denotes the number of hours since the storm began. What is the rate of change of the function d (include units)?. If the storm begins at midnight and is over by 8am, what is the new depth of snow? How much snow did this storm deposit?

$d(t) = 3t + 22$

Rate of Change: 3 inches/hour

Midnight is $t = 0$.

8am is $t = 8$ so $d(8) = 3(8) + 22 = 46$ inches

Deposit $= 46 - D(0) = 46 - 22 = 24$

12. Assume $h \neq 0$ and simplify

$$\frac{(x+h)^3 - x^3}{h}$$

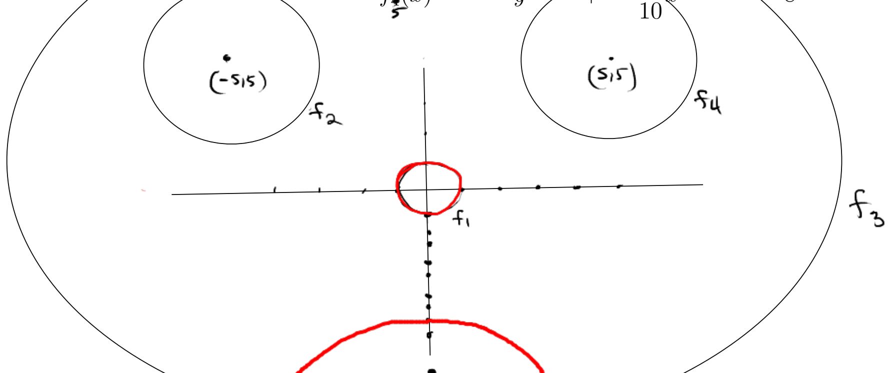
Now let $h = 0$ and evaluate the simplified expression above.

$$\frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h} = \boxed{3x^2 + 3xh + h^2} \quad @\ h=0 \Rightarrow 3x^2$$

13. On the SAME axis plot graphs of the following five functions.

$$\begin{array}{rcccccc} f_1(x) = & x^2 & + & y^2 & = & 1 \\ f_2(x) = & (x+5)^2 & + & (y-5)^2 & = & 9 \\ f_3(x) = & x^2 & + & y^2 & = & 100 \\ f_4(x) = & (x-5)^2 & + & (y-5)^2 & = & 9 \\ f_5(x) = & y & + & \frac{1}{10}x^2 & = & -8 \end{array}$$



14. Without using your calculator, sketch a graph of the following function

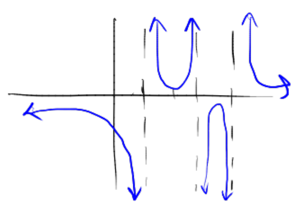
$$f(x) = \frac{1}{x^3 - 8x^2 + 19x - 12} = \frac{1}{(x-1)(x-3)(x-4)}$$

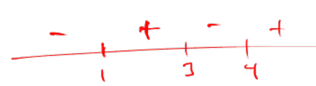
Rational zeros:
$\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

$1^3 - 8 + 19 - 12 = 0$
$\Rightarrow$ $x-1$ is a factor.

zeros: NONE
break points: 1, 3, 4

$$\begin{array}{r|l} & x^2 - 7x + 12 = (x-3)(x-4) \\ x-1 & x^3 - 8x^2 + 19x - 12 \\ & -x^3 - x^2 \\ & -7x^2 \\ & -(-7x^2 + 7x) \\ & 12x - 12 \end{array}$$





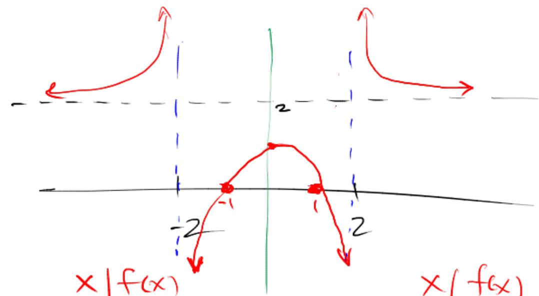
x	$f(x)$
-1000	$1/-1000 < 0$ close to 0
2	$8 - 32 + 38 - 12 = 2 > 0$
4	$1/1000 \to 0$

15. Consider

$$\frac{2(x^2-1)}{x^2-4}$$

Find the zeros of the function.

$x = \pm 1$



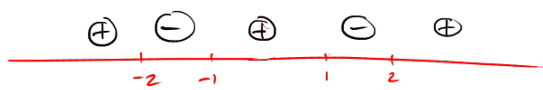
Find the domain of the function.

$\mathbb{R} - \{\pm 2\}$

x	$f(x)$
-100	$\approx \frac{2x^2}{x^2} = 2$
0	$\frac{-2}{-4} = \frac{1}{2}$
100	≈ 2

x	$f(x)$
-1.99	$\approx \frac{2\cdot(4-1)}{-.001} < 0$
1.99	$\approx \frac{2\cdot(4-1)}{} < 0$

Use a number line and test points to find the ~~horizontal~~ asymptotes.
vertical



Find the horizontal asymptotes by dividing each term by the highest power in the denominator letting x be a large number.

$$\frac{2(x^2-1)}{x^2-2} = \frac{2x^2-2}{x^2-2} = \frac{\frac{2x^2}{x^2} - \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{2}{x^2}} = \frac{2 - \frac{2}{x^2}}{1 - \frac{2}{x^2}} \to \frac{2}{1} = 2$$

as $x \to \infty$.

16. Solve for x.

$$e^{(x^2-2x-35)} = 1$$

$\Rightarrow x^2 - 2x - 35 = \ln(1) = 0$

$(x-7)(x+5) = 0 \Rightarrow x = 7, \; x = -5$

17. In 1960 the world (human) population was about 3 billion. By 1974, the population had grown to roughly 4 billion. Using the standard population model

$$P = P_0(1+r)^t$$

where P is the population at time t, P_0 is the inital population and r is the annual growth rate, estimate the annual growth rate of human population during this time frame.

$P_0 = 3$

$P = 3(1+r)^t$

$\Rightarrow 4 = 3(1+r)^{14}$

$\Rightarrow \ln\left(\frac{4}{3}\right) = 14(\ln(1+r))$

$\Rightarrow \frac{1}{14} \cdot \ln\left(\frac{4}{3}\right) = \ln(1+r)$

$r = e^{\left(\frac{1}{14}\ln(4/3)\right)} - 1 = \boxed{.0208} = \boxed{2\%}$

Use your growth rate found above to estimate the world population in 2013. Given that the world population in 2013 is just over 7 billion, evaluate the accuracy of your computation. What are the hidden assumptions that may not have held true during the latter half of the 20^{th} century.

1960 : $t = 0$

2013 : $t = 53$

$\boxed{8.9} = 3(1 + .0208)^{53}$

The growth rate was NOT 2% during the period 1960 – 2013.

$\log\left(\frac{x(x-1)}{2}\right) = 0 \Rightarrow \frac{x(x-1)}{2} = 1$

18. Find all solutions to

$$\log_3(x) + \log_3(x-1) - \log_3(2) = 0$$

$x(x-1) = 2$

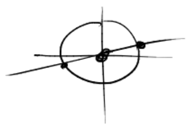
$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\Rightarrow \boxed{x = 2}$

$\Rightarrow x = -1$ (this one is not a solution to original eqn)

19. Compute (exactly) the following



$$\tan\left(\frac{-13\pi}{6}\right) \underline{\hspace{4cm}}$$

$\parallel$ odd

$-\tan\left(\frac{13\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = \frac{-\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}$

same terminal point

$= \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}} = \frac{-1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= -\frac{\sqrt{3}}{3}$

$$\sin\left(\frac{\pi}{2} + \frac{11\pi}{6}\right) \underline{\hspace{4cm}}$$

$\parallel$

$\sin\frac{\pi}{2} \cdot \cos\left(\frac{11\pi}{6}\right) + \cos\left(\frac{\pi}{2}\right) \cdot \sin\left(\frac{11\pi}{6}\right)$

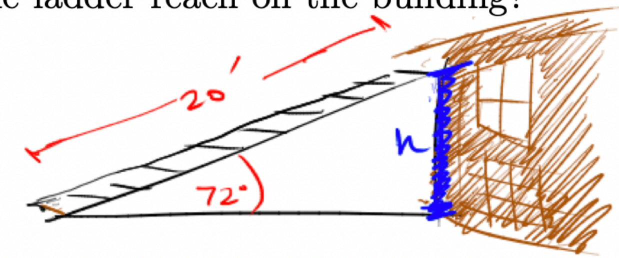
$\sin\frac{\pi}{2} = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$

$\cos\left(\frac{11\pi}{6}\right) = \cos\left(-\frac{\pi}{6}\right)$

$\parallel$ even

$\cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$

#20 A 20-ft ladder leans against a building so that the angle between the ground and the ladder is 72°. How hig does the ladder reach on the building?



$\sin(72^\circ) = \frac{h}{20}$

$h = 20 \sin 72^\circ$

$= \boxed{19'}$

#21

$P(t) = 115 + 25\sin(160\pi t)$

- period $= \frac{2\pi}{160\pi} = \frac{1}{80}$
- amplitude $= 25$
- phase shift: $\emptyset$
- max value: $115 + 25 = 140$

  min value: $115 - 25 = 90$

#22 Verify the identities

$$\cos 2x = \cos^2 x - \sin^2(x) = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$\cos 2x = \cos x \cos x - \sin x \sin x = \boxed{\cos^2 x - \sin^2 x}$

$= (1 - \sin^2 x) - \sin^2 x = \boxed{1 - 2\sin^2 x}$

$= \cos^2 x - (1 - \cos^2 x)$

$= \boxed{2\cos^2 x - 1}$

#23 Verify the identity

$$\sin 2x = 2\sin x \cos x$$

$\sin 2x = \sin(x+x) = \sin x \cos x + \sin x \cos x = 2 \sin x \cos x$

#24 Verify the identity

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} =$$

$$\frac{\sin 2x}{\cos 2x} = \frac{2\sin x \cancel{\cos x} \cdot \frac{1}{\cos^{\cancel{2}} x}}{(\cos^2 x - \sin^2 x)\frac{1}{\cos^2 x}} = \frac{\frac{2\sin x}{\cos x}}{1 - \tan^2 x} = \boxed{\frac{2\tan x}{1-\tan^2 x}}$$

#25 Verify the identity

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

start:

$\cos 2x = 2\cos^2 x - 1$ (#23 above)

$\downarrow$

$\cos 2x + 1 = 2\cos^2 x$

$\downarrow$

$\frac{\cos 2x + 1}{2} = \cos^2 x$

#26 Verify the identity

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

start:

$\cos 2x = 1 - 2\sin^2 x$ #(24 above)

swap

$2\sin^2 x = 1 - \cos 2x \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

#27 Verify the identity

$$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$\sin a \underbrace{\cos(-b)}_{\cos b} + \underbrace{\sin(-b)}_{-\sin b} \cos a$

$\sin a \cos b + \sin b \cos a$

$= \frac{1}{2}(2 \sin a \cos b) = \sin a \cos b$

this one is out of order!

#28 Find all solutions to

$\boxed{x = n \cdot \pi \text{ where } n \text{ is an integer}}$

$$\sin x = 0$$

$$\cos(2x) = 0$$

this equation forces $2x = \pm\frac{\pi}{2} + 2n\pi$

$\frac{3\pi}{4} \cdot 2 = \frac{3\pi}{2}$ $(\cos = 0)$ ✓

$\frac{5\pi}{4} \times 2 = \frac{5\pi}{2} \leftrightarrow \frac{\pi}{2}$ $(\cos = 0)$ ✓

*doubles

$\frac{7\pi}{4} \cdot 2 = \frac{7\pi}{2} \leftrightarrow \frac{3\pi}{2}$ $(\cos = 0)$ ✓

so $\boxed{x = \pm\frac{\pi}{4} + n\pi}$

#28

29. Solve $\tan^2 x - 1 = 0$.

$$\tan^2 x = 1$$
$$\tan x = \pm 1$$

$\tan x = 1$

$\Rightarrow x = \frac{\pi}{4} + n\pi$

$\tan x = -1$

$x = -\frac{\pi}{4} + n\pi$

$$\boxed{x = \pm\frac{\pi}{4} + n\pi}$$

30. Solve $\sin^2\theta = 2\sin\theta + 3$.

quadratic type

$$\sin^2\theta - 2\sin\theta - 3 = 0$$
$$(\sin\theta - 3)(\sin\theta + 1) = 0$$

$\sin\theta = 3$ or $\sin\theta = -1$

no solutions

$\boxed{\theta = -\frac{\pi}{2} + 2n\pi}$

use 2π because that's the period of sin

31. Solve $2\cos^2\theta - 2\sqrt{2}\cos\theta - 1$.

quadratic type... we can use quadratic formula to solve for $\cos\theta$

$a = 2$, $b = -2\sqrt{2}$, $c = -1$

$$\cos\theta = \frac{2\sqrt{2} \pm \sqrt{8 + 4 \cdot 2 \cdot 1}}{2 \cdot 2}$$
$$= \frac{2\sqrt{2} \pm \sqrt{16}}{4}$$
$$= \frac{2\sqrt{2} \pm 4}{4} = \frac{\sqrt{2}}{2} \pm 1$$

$\cos\theta = \frac{\sqrt{2}}{2} \pm 1$

$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{2} + 1\right)$ (DNE!)

$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2} - 1\right)$

$\boxed{= 1.86 \text{ radians or } 107.03^\circ}$

32. Solve $3\tan^3 x = \tan x$.

$3\tan^3 x - \tan x = 0 =$ tan(x) (3*tan(x)^2 – 1) = 0

so either:

tan(x) = 0 which implies sin(x) = 0 so $\boxed{x = n*pi}$

or tan(x) = sqrt(1 / 3) = sqrt(3) / 3 so $\boxed{x = pi/6 + pi}$

or tan(x) = -sqrt(1 / 3) = -sqrt(3)/3 so $\boxed{x = 11pi/6 + pi}$

33. If $\sec t = 11/6$ and the terminal point of t is in quadrant IV, find

$\frac{1}{\cos(t)} = \frac{11}{6}$

$\Rightarrow \cos(t) = \frac{6}{11}$

so $\cos^2(t) + \sin^2(t) = 1 \Rightarrow \frac{36}{121} + \sin^2(t) = 1$

$\Rightarrow \sin^2(t) = \frac{121}{121} - \frac{36}{121} = \frac{85}{121} \Rightarrow \sin(t) = \pm\sqrt{\frac{85}{121}}$

Q IV $\Rightarrow \sin(t) < 0$ so choose $-\sqrt{85}/121 = -\frac{\sqrt{85}}{11}$

$\sin t =$ $-\frac{\sqrt{85}}{11}$

$\cos t =$ $6/11$

$\tan t =$ $-\sqrt{85}/6$

$\cot t =$ $\frac{-6}{\sqrt{85}}$

$\csc t =$ $\frac{-11}{\sqrt{85}}$

34. **Terminal Point**

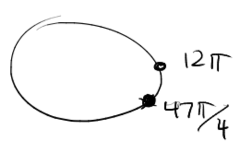
Find the terminal point of $13\pi/3$?

$\frac{13\pi}{3} = 4\pi + \frac{\pi}{3}$

$T.P\left(\frac{13\pi}{3}\right) = T.P.\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Find the terminal point of $47\pi/4$?

$\frac{47\pi}{4} = \frac{48\pi}{4} - \frac{\pi}{4}$ ($\frac{48\pi}{4} = 12\pi$)

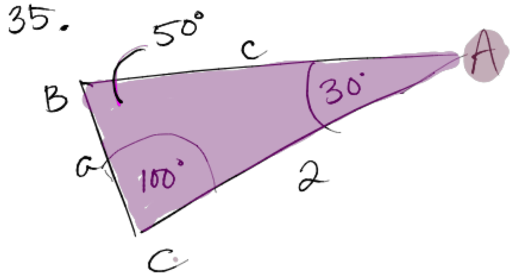


reference # : $\frac{\pi}{4}$

quadrant : IV

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

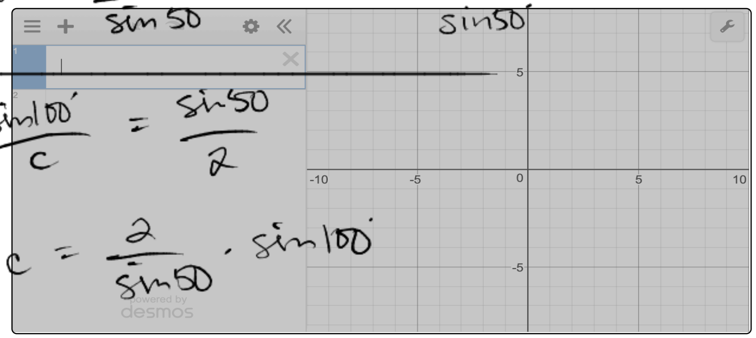
35.



ASA ⇒
Law of sines

$$\frac{\sin 50^\circ}{2} = \frac{\sin 30}{a}$$

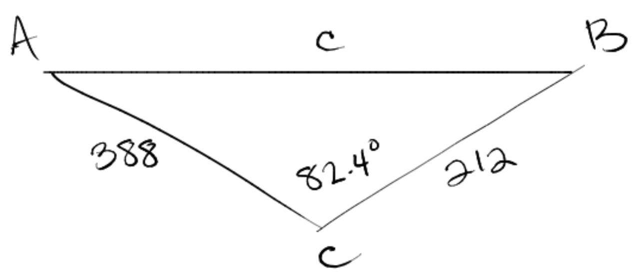
$$a = \frac{2}{\sin 50} \cdot \sin 30 = \frac{1}{\sin 50}$$



$$\frac{\sin 100^\circ}{c} = \frac{\sin 50}{2}$$

$$c = \frac{2}{\sin 50} \cdot \sin 100^\circ$$

36.



SAS - Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

$$= 212^2 + 388^2 - 2 \cdot 212 \cdot 388 \cdot \cos 82^\circ$$

$$\approx 415^\circ$$

(smallest angle first!)

$$\frac{\sin A}{212} = \frac{\sin 82^\circ}{415}$$

$$A = \sin^{-1}\left(\frac{212}{415} \cdot \sin 82^\circ\right)$$

$$A \approx 30.4^\circ$$

$$B \approx 67.2^\circ$$

And by the way ...