Applications

1. Optimization

(a) A rectangular storage container with a closed top and square base is to have a volume of 9 cubic meters. Material for the base costs \$3 per square meter. Material for the sides costs \$6 per square meter. Material for the top costs \$1 per square meter.

Find the cost and dimensions of the cheapest such container.

Constraint (relationship between x and y): $x^2y = 9 \longrightarrow y = 9/x^2$
Expression to maximize (c0st): $3x^2 + 4(6xy) + 1x^2 = 4x^2 + 24xy$
Put the expression in terms of a single variable:

$$f(x) = 4x^2 + 24x\left(\frac{9}{x^2}\right) = 4x^2 + \frac{216}{x}$$

$$f'(x) = 8x - \frac{216}{x^2} \overset{\text{set}}{=} 0 \Longrightarrow x = 3 \rightarrow y = 1$$

Dioes this value of x reallly minimize cost? Use the second derivative test to check:

$$f''(x) = 8 + \frac{432}{x^3} \longrightarrow f''(3) = 24 > 0$$

Since $f''(3) > 0$, the graph of the cost function is concave up at $x = 3$, so we did minimize cost.
Answer: dimensions $3 \times 3 \times 1$

(b) Find the point on the line $y = 2x + 5$ that is closest to the point (2,0). Draw a sketch.

Distance from a point on the line to the point (2,0) is given by

$$D(x) = \sqrt{(x-2)^2 + (2x+5-0)^2} = \sqrt{5x^2 + 16x + 29}$$

Take derivative:

$$D'(x) = \frac{1}{2}\left(5x^2 + 16x + 29\right)^{-1/2}(10x + 16) = \frac{5x + 8}{\sqrt{5x^2 + 16x + 29}}$$

Set the derivative equal to zero and solve for x:

$$\frac{5x + 8}{\sqrt{5x^2 + 16x + 29}} = 0 \Longrightarrow x = -1.6 \rightarrow y = 1.8$$

The sketch demonstrates that this point does minimize the distance - no need to use second derivative test here.

2. Find the volume of the solid obtained by revolving the region between the x-axis and the curve $y = 1/x$, $1 \le x \le 4$, around the x-axis.

$$\int_1^4 \pi \left(\frac{1}{x}\right)^2 dx = \int_1^4 \pi x^{-2}\, dx = \left[-\frac{\pi}{x}\right]_1^4 = -\frac{\pi}{4} - \left(-\frac{\pi}{1}\right) = \frac{3}{4}\pi$$

3. Find the work done when a particle is moved along the x-axis from the origin to $x = 5$ by a force given by $f(x) = 2e^{-2x} + 1$.

$$\text{Work } = \int_0^5 2e^{-2x}+1\, dx = \left[-e^{-2x} + x\right]_0^5 = \left(-e^{-10} + 5\right) - (-1 + 0) = 6 - e^{-10} J$$

4. Given the following, find $f(x)$.
(a) $f(x) = 6x - 2$
(b) $f'(2) = 5$
(c) $f(1) = 3$

$$f'(x) = \int 6x - 2\, dx = 3x^2 - 2x + C$$

$$5 = f'(2) = 3(2)^2 - 2(2) + C \longrightarrow C = -3$$

$$f(x) = \int 3x^2 - 2x - 3\, dx = x^3 - x^2 - 3x + C$$

$$3 = f(1) = (1)^3 - (1)^2 - 3(1) + C \longrightarrow C = 6$$

Answer: $f(x) = x^3 - x^2 - 3x + 6$

5. Use a linear approximation, $L(x) = f(a) + f'(a)(x - a)$, to estimate $(1.05)^4$.

First find the tangent line to the function $f(x) = x^4$ at $x = 1$.
Point on the line: (1,1)
Slope of tangent line at $x = 1$: $f'(1) = 4(1)^3 = 4$
Tangent line to $y = x^4$ at $x = 1$: $y = 4x - 3$

Now evaluate the tangent line when $x = 1.05$:

$$(1.05)^4 \approx 4(1.05) - 3 = 1.20$$