

Antiderivatives 3

Find the indicated antiderivative. Check your answers.

1. Use $u = \tan^{-1} x \rightarrow du = \frac{1}{x^2+1} dx \rightarrow (x^2 + 1) du = dx$

$$\int \frac{(\tan^{-1} x)^3}{x^2 + 1} dx = \int \frac{(u)^3}{x^2 + 1} (x^2 + 1) du = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}(\tan^{-1} x)^4 + C$$

Check

$$\frac{d}{dx} \left[\frac{1}{4} (\tan^{-1} x)^4 \right] = \frac{4}{4} (\tan^{-1} x)^3 \frac{d}{dx} [\tan^{-1} x] = (\tan^{-1} x)^3 \left(\frac{1}{x^2 + 1} \right) = \frac{(\tan^{-1} x)^3}{x^2 + 1}$$

2. Use $u = x^2 + 1 \rightarrow du = 2x dx \rightarrow \frac{1}{2x} du = dx$

$$\int \frac{4x}{x^2 + 1} dx = \int \frac{4x}{u} \cdot \frac{1}{2x} du = \int \frac{2}{u} du = 2 \int \frac{1}{u} du = 2 \ln |u| + C = 2 \ln (x^2 + 1) + C$$

Check:

$$\frac{d}{dx} [2 \ln (x^2 + 1)] = 2 \left(\frac{\frac{d}{dx} [x^2 + 1]}{x^2 + 1} \right) = \frac{2(2x)}{x^2 + 1} = \frac{4x}{x^2 + 1}$$

3. Use $u = x^2 + 1 \rightarrow du = 2x dx \rightarrow \frac{1}{2x} du = dx$

$$\int \frac{4x}{(x^2 + 1)^2} dx = \int \frac{4x}{(u)^2} \cdot \frac{1}{2x} du = \int \frac{2}{u^2} du = \int 2u^{-2} du = 2 \frac{u^{-1}}{-1} + C = -\frac{2}{x^2 + 1} + C$$

Check:

$$\frac{d}{dx} \left[-\frac{2}{x^2 + 1} \right] = \frac{d}{dx} \left[-2(x^2 + 1)^{-1} \right] = -2(-1)(x^2 + 1)^{-2} \frac{d}{dx} [x^2 + 1] = \frac{2(2x)}{(x^2 + 1)^2}$$

4.

$$\int \frac{5}{x^2 + 1} dx = 5 \int \frac{1}{x^2 + 1} dx = 5 \tan^{-1} x + C$$

Check:

$$\frac{d}{dx} [5 \tan^{-1} x] = 5 \cdot \frac{1}{x^2 + 1} = \frac{5}{x^2 + 1}$$

5. Use #2 and #4 above.

$$\int \frac{4x + 5}{x^2 + 1} dx = \int \frac{4x}{x^2 + 1} + \frac{5}{x^2 + 1} dx = 2 \ln (x^2 + 1) + 5 \tan^{-1} x + C$$

6. Use $u = 1 - x^2 \rightarrow du = -2x dx \rightarrow -\frac{1}{2x} du = dx$

$$\begin{aligned}\int x\sqrt{1-x^2} dx &= \int x\sqrt{u} \left(-\frac{1}{2x}\right) du = \int -\frac{1}{2}u^{1/2} du = -\frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C \\ &= -\frac{1}{2} \cdot \frac{2}{3}u^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C\end{aligned}$$

Check:

$$\frac{d}{dx} \left[-\frac{1}{3}(1-x^2)^{3/2} \right] = -\frac{1}{3} \cdot \frac{3}{2} (1-x^2)^{1/2} \frac{d}{dx} [1-x^2] = -\frac{1}{2} \sqrt{1-x^2} (-2x) = x\sqrt{1-x^2}$$

7. Use $u = 1 - x^2 \rightarrow du = -2x dx \rightarrow -\frac{1}{2x} du = dx$

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}} dx &= \int \frac{x}{\sqrt{u}} \left(-\frac{1}{2x}\right) du = \int -\frac{1}{2}u^{-1/2} du = -\frac{1}{2} \cdot \frac{u^{1/2}}{1/2} + C \\ &= -\frac{1}{2} \cdot \frac{2}{1} \sqrt{1-x^2} + C = -\sqrt{1-x^2} + C\end{aligned}$$

Check:

$$\frac{d}{dx} [-\sqrt{1-x^2}] = -\left(\frac{1}{2}\right) (1-x^2)^{-1/2} \frac{d}{dx} [1-x^2] = -\frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} (-2x) = \frac{x}{\sqrt{1-x^2}}$$

8. Use $u = \sin^{-1} x \rightarrow du = \frac{1}{\sqrt{1-x^2}} dx \rightarrow \sqrt{1-x^2} du = dx$

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int \frac{u}{\sqrt{1-x^2}} (\sqrt{1-x^2}) du = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sin^{-1} x)^2 + C$$

Check:

$$\frac{d}{dx} \left[\frac{1}{2}(\sin^{-1} x)^2 \right] = \frac{1}{2}(2)(\sin^{-1} x)^1 \frac{d}{dx} [\sin^{-1} x] = \sin^{-1} x \left(\frac{1}{\sqrt{1-x^2}} \right) = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

9.

$$\int \frac{3}{\sqrt{1-x^2}} dx = 3 \int \frac{1}{\sqrt{1-x^2}} dx = 3 \sin^{-1} x + C$$

Check:

$$\frac{d}{dx} [3 \sin^{-1} x] = 3 \left(\frac{1}{\sqrt{1-x^2}} \right) = \frac{3}{\sqrt{1-x^2}}$$

10. Use #7 and #9

$$\int \frac{x-3}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} - \frac{3}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} - 3 \sin^{-1} x + C$$