

Name: KEY

Find the indicated antiderivative.

$$1. \int \frac{3x^2}{\sqrt{x^3 - 1}} dx = \int u^{-1/2} du = \frac{u^{1/2}}{\frac{1}{2}} + C$$

$$u = x^3 - 1$$

$$du = 3x^2 dx$$

$$= \boxed{2\sqrt{x^3 - 1} + C}$$

$$\text{check: } \frac{d}{dx}(\text{ans}) = \frac{2}{2}(x^3 - 1)^{-1/2} \cdot 3x^2 = \frac{3x^2}{\sqrt{x^3 - 1}} \quad \checkmark$$

Exam 3 STUDY GUIDE
CHECK YOUR ANSWERS



$$2. \int \frac{e^{\cot(x)} \csc(x)}{\sin(x)} dx =$$

(1) Algebra rewrite: $(\frac{1}{\sin x} = \csc x)$

$$= \int e^{\cot(x)} \cdot \csc^2 x dx = - \int e^{\cot x} (-\csc^2 x) dx = - \int e^u du = \boxed{-e^{\cot x} + C}$$

$$\text{so } u = \cot x$$

$$du = -\csc^2 x dx$$

check:

$$\begin{aligned} \frac{d}{dx}(-e^{\cot x} + C) &= -e^{\cot x} \cdot (-\csc^2 x) \\ &= e^{\cot x} \cdot \csc x \cdot \frac{1}{\sin x} = \frac{e^{\cot x} \csc x}{\sin x} \quad \checkmark \end{aligned}$$

$$3. \int 6x^2 \sin(x^3) \cos(x^3) dx =$$

Here there are two tempting choices for

$$u = x^3 \quad \frac{1}{2} \underbrace{u = \sin(x^3)}$$

better choice!

$$u = \sin(x^3)$$

$$du = \cos(x^3) \cdot 3x^2 dx$$

so

$$2du = \cos(x^3) \cdot 6x^2 dx$$

$$\frac{d}{dx}((\sin(x^3))^2 + C)$$

$$= 2(\sin(x^3)) \cdot \cos(x^3) \cdot 3x^2$$

/ double chain rule!

equals integrand!
($2 \cdot 3 = 6$)

$$\text{(Sub)} \quad = \int u \cdot 2du = 2u^2 + C = u^2 + C = \boxed{[\sin(x^3)]^2 + C}$$

$$4. \int \frac{1}{(\sin^{-1} x)^5 \sqrt{1-x^2}} dx = \int u^5 du = \frac{u^6}{6} + C = -\frac{1}{4} \cdot (\sin^{-1} x)^4 + C$$

$$u = \sin^{-1} x \\ du = \frac{1}{\sqrt{1-x^2}} dx \\ \frac{d}{dx}(\text{ans}) = \frac{(-4)(\sin^{-1} x)^{-5}}{\sqrt{1-x^2}} \quad \checkmark$$

this one comes out in front

$$5. \int \frac{4x}{x^2+1} dx = 2 \int \frac{2x}{x^2+1} dx = 2 \int \frac{du}{u} = 2 \ln|u| + C = 2 \ln|x^2+1| + C$$

$$u = x^2+1 \\ du = 2x dx \\ \left(\text{ignore abs value since } x^2+1 > 0 \right)$$

$$\frac{d}{dx}(\text{ans}) = 2 \frac{(2x)}{x^2+1} = \frac{4x}{x^2+1} \quad \checkmark$$

$$6. \int \frac{2-4x}{x^2+1} dx = \int \frac{2}{x^2+1} dx - \int \frac{4x}{x^2+1} dx \\ = 2 \tan^{-1} x - 2 \ln|x^2+1| + C$$

$$\left| \frac{d}{dx}(\text{ans}) = \frac{2}{x^2+1} - \frac{4x}{x^2+1} \right. \quad \text{(above)}$$

$$7. \int x \sqrt{2x-1} dx = \int x \sqrt{u} \frac{1}{2} du \quad \text{(see below)} \\ \left| \begin{array}{l} u = 2x-1 \\ du = 2 dx \\ \frac{1}{2} du = dx \end{array} \right. \quad \text{mixed variables do rewrite } x \text{ in terms of } u.$$

$$\frac{1}{2}(u+1) = x$$

$$\begin{aligned} & \text{back it up by 1} \not\rightarrow \text{divide by 4} \\ & = \frac{1}{4} \left[\frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right] + C \\ & = \frac{1}{4} \left[\frac{2}{5} (2x-1)^{5/2} + \frac{2}{3} (2x-1)^{3/2} \right] + C \end{aligned}$$

check

$$\begin{aligned} \frac{d}{dx}(\text{ans}) &= \frac{1}{4} \left[\frac{5}{2} \frac{2}{5} (2x-1)^{3/2} \cdot 2 + \frac{3}{2} \frac{2}{3} (2x-1)^{1/2} \cdot 2 \right] \\ &= \frac{1}{2} \left[(2x-1)^{3/2} + (2x-1)^{1/2} \right] \\ &= \frac{1}{2} \left[(2x-1)^{\frac{1}{2}} \left[(2x-1)^{\frac{1}{2}} + 1 \right] \right] \\ &= \frac{1}{2} \left[(2x-1)^{\frac{1}{2}} (2x-1 + 1) \right] \\ &= \frac{1}{2} ((2x-1)^{\frac{1}{2}} (2x)) \\ &= x (2x-1)^{1/2} \quad \checkmark \end{aligned}$$

$$8. \int \frac{4}{x\sqrt{x^8-1}} dx = \int \frac{4}{x\sqrt{u^2-1}} \cdot \frac{1}{4x^3} du = \int \frac{1}{x^4\sqrt{u^2-1}} du = \int \frac{1}{u\sqrt{u^2-1}} du$$

Inverse trig:

$$u = x^4 \Rightarrow u^2 = x^8$$

$$du = 4x^3 dx$$

$$\frac{1}{4x^3} du = dx$$

Check

$$\frac{d}{dx}(\text{ans}) = \frac{1}{x^4\sqrt{(x^4)^2-1}} \cdot 4x^3 = \frac{4}{x^4\sqrt{x^8-1}}$$

$$= \sec^{-1}(u) + C$$

$$= \boxed{\sec^{-1}(x^4) + C}$$

$$9. \int 7x^3\sqrt{x^4-1} dx = 7 \int x^3 \sqrt{u} \frac{1}{4x^3} du = \frac{7}{4} \int u^{\frac{1}{2}} du = \frac{7}{4} \underbrace{u^{\frac{3}{2}}}_{\frac{3}{2}} = \frac{2}{3} \cdot \frac{7}{4} u^{\frac{3}{2}} + C$$

$$u = x^4 - 1$$

$$du = 4x^3 dx$$

$$\frac{1}{4x^3} du = dx$$

Check

$$\frac{d}{dx}(\text{ans}) = \frac{3 \cdot 7}{2 \cdot 6} \cdot (x^4 - 1)^{\frac{1}{2}} \cdot 4x^3$$

$$= \frac{7}{4} (x^4 - 1)^{\frac{1}{2}} \cdot 4x^3 = 7x \cdot (x^4 - 1)^{\frac{1}{2}}$$

$$= \boxed{\frac{7}{6} (x^4 - 1)^{\frac{3}{2}} + C}$$

$$10. \int \frac{x^2}{\sqrt{1-x^6}} dx \quad \text{no derivative relationships, fractions, involving radicals} \Rightarrow$$

- think inverse trig

$$u = x^3$$

$$du = 3x^2 dx$$

$$\text{also: } u^2 = x^6$$

\downarrow multiply & divide by 3 to match du

$$\frac{1}{3} \int \frac{3x^2}{\sqrt{1-u^2}} du \quad \text{matches du !!!} \quad = \frac{1}{3} \int \frac{du}{\sqrt{1-u^2}} = \boxed{\frac{1}{3} \sin^{-1}(x^3) + C}$$

Check

$$\frac{d}{dx}(\text{ans}) = \frac{1}{3} \cdot \left(\frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2 \right) = \frac{x^2}{\sqrt{1-x^6}} \quad \checkmark$$

$$11. \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int \frac{e^u}{\sqrt{x}} \cdot 2\sqrt{x} du = 2 \int e^u du = 2e^{\sqrt{x}} + C$$

$$u = \sqrt{x}$$

$$du = \frac{1}{2} x^{-\frac{1}{2}} dx$$

$$2x^{\frac{1}{2}} du = dx$$

Check:

$$\frac{d}{dx}(\text{ans}) = 2e^{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}$$

$$12. \int \tan^2(x) \sec^2(x) dx = \int u^2 du = \frac{u^3}{3} + C = \left(\frac{\tan x}{3}\right)^3 + C$$

derivative relationship: $\frac{d}{dx}(\tan x) = \sec^2 x$!

$$u = \tan x \\ du = \sec^2 x dx$$

$$\frac{d}{dx}(\text{ans}) = \frac{3}{3} (\tan x)^2 \cdot \sec^2 x = \tan^2 x \cdot \sec^2 x \quad \checkmark$$

should be absolute value $\frac{1}{x \cdot \ln(3x)}$

$$13. \int \frac{1}{x \ln(3x)} dx =$$

most complicated expression: $\ln(3x)$

$$u = \ln(3x)$$

$$du = \frac{1}{3x} dx = \frac{1}{x} dx \\ x du = dx$$

$$\text{check} \\ \frac{d}{dx}(\text{ans}) = \frac{d}{dx}(\ln(3x)) = \frac{1}{x} = \frac{1}{x \cdot \ln(3x)} + C$$

$$\left. \begin{array}{l} \text{sub} \\ \int \frac{1}{x \cdot u} \cdot x du = \int \frac{du}{u} = \ln|u| + C \\ = \ln|\ln(3x)| + C \end{array} \right\}$$

$$14. \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{\cos x}{(1+\sin x)^{1/2}} dx = \int \frac{du}{(u)^{1/2}} = \int u^{-\frac{1}{2}} du = \frac{2}{1} u^{\frac{1}{2}} + C = 2(1+\sin x)^{\frac{1}{2}} + C$$

$$u = 1 + \sin x \quad (\text{inside of parenthesis})$$

$$du = \cos x dx$$

matches exactly with part of integrand, so we can substitute.

check

$$\frac{d}{dx}(\text{ans}) = \frac{2}{2}(1+\sin x)^{-\frac{1}{2}} \cdot \cos x = \frac{\cos x}{\sqrt{1+\sin x}}$$

$$15. \int \frac{\sin(e^{\cos^{-1} x}) e^{\cos^{-1} x}}{\sqrt{1-x^2}} dx =$$

The idea here is to see a "chain" of derivative relationships!

$$\text{set } u = e^{\cos^{-1} x}$$

$$du = e^{\cos^{-1} x} \cdot \frac{-1}{\sqrt{1-x^2}} dx \quad \text{so } dx = \frac{-\sqrt{1-x^2}}{e^{\cos^{-1} x}} du$$

$$\left. \begin{array}{l} \text{sub} \\ \int \frac{\sin(u) e^{\cos x}}{\sqrt{1-x^2}} \cdot \frac{-\sqrt{1-x^2}}{e^{\cos^{-1} x}} du \\ = - \int \sin(u) du = \cos(u) + C \\ = \cos(e^{\cos^{-1} x}) + C \end{array} \right\} \text{this remains!}$$

Check: 1st set $w = e^{\cos^{-1} x}$, note $\frac{dw}{dx} = e^{\cos^{-1} x} \frac{-1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}(\text{ans}) \\ =$$

$$\frac{d}{dx}(\cos(w)) = -\sin(w) \cdot \frac{dw}{dx} = -\sin(w) \cdot e^{\cos^{-1} x} \cdot \frac{-1}{\sqrt{1-x^2}} = \frac{\sin(e^{\cos^{-1} x}) \cdot e^{\cos^{-1} x}}{\sqrt{1-x^2}}$$