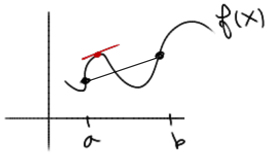
The Fundamental Theorem of Calculus

Key Tool: Mean-Value theorem



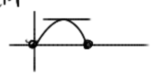
For a cts, differentiable function $f(x)$ on $[a,b]$ —

$\exists\ c \in [a,b]$ s.t. $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

or

$$f(b)-f(a) = f'(c)(b-a)$$

EX displacement



start @ home,
end @ home
$\exists$ some pt. where derivative $= 0$

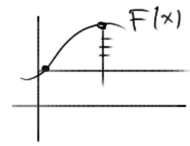
Theorem $\int_a^b f(x)\,dx = F(b) - F(a) \overset{2}{=} F(b) \underbrace{- F(x_1) + F(x_1)}_{=0} \underbrace{- F(x_2) + F(x_2)}_{=0} \underbrace{- F(x_3) + \dots}_{=0} - F(a)$

1 $\parallel$

$\lim_{n\to\infty} \sum_{i=0}^{n} \Delta x \cdot f(x_i^*)$

$\overset{3}{=} \sum_{i=0}^{k} \underbrace{F(x_i) - F(x_{i+1})}_{}$

apply M.V.T. $\Rightarrow \exists\ x_i^*$ s.t.



$\overset{4}{=} \sum_{i=0}^{k} f'(x_i^*) \cdot (x_i - x_{i+1})$

take limit as $k \to \infty$, note: $\Delta x = x_i - x_{i+1}$

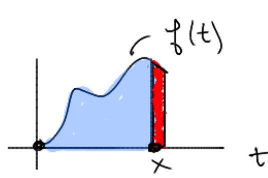
$\overset{5}{=} \lim_{k\to\infty} \sum_{i=0}^{k} f'(x_i^*) \cdot (x_i - x_{i+1})$

$\overset{6}{=} \int_a^b f(x)\,dx$

Surprising link b/w  and 

I $\quad \underbrace{\int_a^b f(x)\,dx}_{\text{area}} = F(b) - F(a) \quad$ w/ $\frac{d}{dx}(F(x)) = f(x)$

II Area so far:



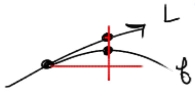
Shaded $= \int_0^x f(t)\,dt = A(x)$

$A'(x) = \dfrac{\text{new total area} - \text{old total area}}{\text{width}} \approx f(t)$ height

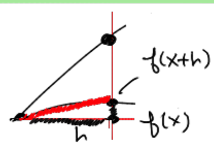
$A'(x) = \boxed{\dfrac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)}$

Part II

Recall, $f'(x) = \dfrac{f(x+h) - f(x)}{h}$

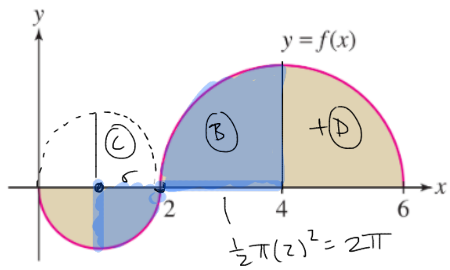


zoom scale



Evaluate the two integrals given the graph. The two parts of the graph are semicircles.

$\int_1^4 f(x)\,dx$ and $\int_1^6 |f(x)|\,dx = \left|\frac{-\pi}{4}\right| + 2\pi$



$\frac{1}{2}\pi(2)^2 = 2\pi$

$$\int_1^4 f(x)\,dx = \underbrace{\int_1^2 f(x)\,dx}_{A} + \underbrace{\int_2^4 f(x)\,dx}_{B} = \frac{-\pi}{4} + \pi = \frac{3\pi}{4}$$

A

Full Circle: πr^2
$r = 1$ 〃 π

Area. Semi-Circle $= \frac{\pi}{2}$

Quarter Circle
$\pi/4 \Rightarrow A = \frac{-\pi}{4}$

B

Full Circle . πr^2
$r = 2 \Rightarrow \pi \cdot 4 = 4\pi$

B $= 1/4$ circle w/ $r = 2$

B $= \pi$

(Give an exact answer. Use symbolic notation and fractions where needed.)

$\int_1^4 f(x)\,dx =$ ______

5.3.8 Keys to Integrals: integrand must match some known form (chart)
[adjust integrand to match]
- algebra
- break it properly

$$\int \frac{34x^3 + 42x - 67}{x^2}\, dx =$$

Basic Integrals Chart

Function	Anti-derivative	Function
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{1}{1+x^2}$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{1}{\sqrt{1-x^2}}$
e^x	e^x	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\sin(x)$	$-\cos(x)$	
$\cos(x)$	$\sin(x)$	
$\sec(x)\tan(x)$	$\sec(x)$	
$\sec^2(x)$	$\tan(x)$	

$$= \int \frac{34x^3}{x^2} + \frac{42x}{x^2} - \frac{67}{x^2}\, dx$$

$$= \int 34x + \frac{42}{x} - \frac{67}{x^2}\, dx$$

$$34\int x\, dx + 42\int \frac{1}{x}\, dx - 67\int \frac{1}{x^2}\, dx$$

$$34\int x\, dx + 42\int \frac{1}{x}\, dx - 67\int x^{-2}\, dx = 17x^2 + 42\ln(\mathrm{abs}(x)) + \frac{67x^{-1}}{1} + C$$

The Fundamental theorem of Calculus

(as in, deep, (not simple))

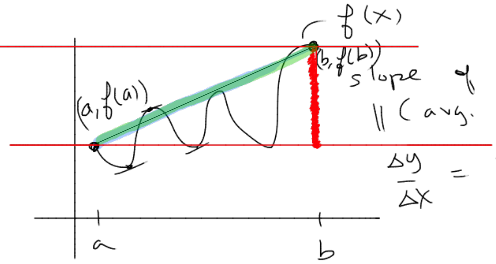
Key Tool: Mean Value Theorem
(average)

Idea: How many miles b/w MQT & Houghton? ~ 100.
Suppose you drive there in one hour.

Your speed may vary throughout.

MVT => there's at least one time, where your speedometer reads EXACTLY 100 m/h.

Average rate of change is achieved somewhere along the way.



slope of secant line || (avg. rate of change)

$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b-a}$

(assumption: $f(x)$ must be continuous / differentiable)

M.V.T there is some $x \in (a,b)$

st $f'(x) = \frac{f(b)-f(a)}{b-a}$

or $\boxed{f(b) - f(a) = f'(x)[b-a]}$

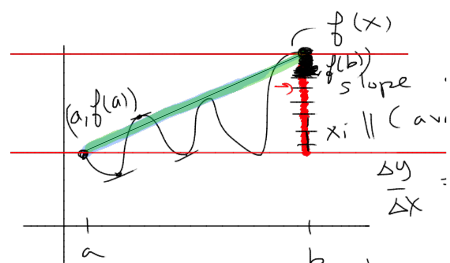
Fund. thm of Calc.

w/ F = some anti-derivative of $f(x)$

(part I) $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

def'n

$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x$



$= F(\underset{x_{-1}}{b}) - F(x_0) + F(x_0) - F(x_1) + F(x_1) - F(x_2) + \dots - F(a)$

$= \sum_{i=-1}^{k} \underbrace{F(x_i) - F(x_{i+1})}_{\text{apply M.V.T.}}$

$\Delta x = x_i - x_{i+1}$

$= \sum_{i=-1}^{k} F'(x^*)\Delta x_i$

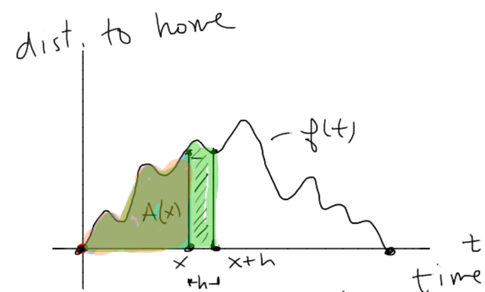
now take $\lim_{k\to\infty}$

$= \lim_{k\to\infty} \sum_{i=-1}^{k} F'(x^*)\Delta x_i = \int_a^b f(x)\,dx$

part II

Area so far function:

$$A(x) = \int_0^x f(t)\,dt$$

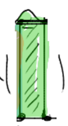


$A(x)$ = total dist traveled.

$$\frac{d}{dx}\big(A(x)\big) = A'(x) = \underline{\quad f(x) \quad}$$

Recall

$$A'(x) \approx \frac{A(x+h) - A(x)}{h} = \frac{\text{Area}(\quad)}{h}$$



$$\approx \frac{\overset{f(x)}{\text{height}} \times \overset{h}{\text{width}}}{h} \approx \frac{f(x)\cdot h}{h}$$

take limit $= f(x)$

Ex

$$A(x) = \int_0^x \cos(t)\,dt \quad , \quad A'(x)$$

$A'(\frac{\pi}{2})$

$A'(2\pi)$