Initial Value Problems

given

9. $f'(x) = (x^2-1)/x$, $f(1) = \frac{1}{2}$

eg. speed   position @ time = 1

couple $\rightsquigarrow \int \Rightarrow$ full description of position

10. $f''(x) = 4-6x-40x^3$, $f'(0) = 1$, $f(0) = 2$

eg. acceleration   velocity @ one instant   position @ one instant

integrate all this

$\Rightarrow$ complete description of position

$f(x) = \int f'(x)\,dx$ & $f'(x) = \int \underbrace{f''(x)}_{\text{start}}\,dx$

$f'(x) = \int 4-6x-40x^3\,dx = 4x - \frac{6x^2}{2} - \frac{40x^4}{4} + C$

use $f'(0) = 1 = 4\cdot 0 - \frac{6\cdot 0^2}{2} - \frac{4\cdot 0^4}{4} + C \Rightarrow \boxed{C=1}$

update

$f'(x) = 4x - 3x^2 - 10x^4 + 1$ repeat

$f(x) = \int f'(x)\,dx = \int 4x - 3x^2 - 10x^4 + 1\,dx$

$= \frac{4x^2}{2} - \frac{3x^3}{3} - \frac{10x^5}{5} + x + D$

Find D via $f(0) = 2$

$\underset{\overset{\shortparallel}{2}}{f(0)} = 4\cdot 0 - 3\cdot 0 - 10\cdot 0 + 0 + D$

$\Rightarrow D = 2$

clean & update

$\boxed{f(x) = 2x^2 - x^3 - 2x^5 + x + 2}$

Idea $f(t) = \int f'(t)\,dt$

① $f(x) = \int \frac{x^2-1}{x}\,dx \overset{\text{algebra 1st}}{=} \int x - \frac{1}{x}\,dx = \frac{x^2}{2} - \ln|x| + C$

no prime

② sub $x=1$

$\underset{\overset{\shortparallel}{1/2}}{f(1)} = \frac{1^2}{2} - \ln|1| + C = \frac{1}{2} - \underset{\overset{\shortparallel}{0}}{\ln|1|} + C$

$\frac{1}{2} = \frac{1}{2} - 0 + C$

$C = 0$

③ $f(x) = \frac{x^2}{2} - \ln|x|$

check: $f'(x) = \frac{2x}{2} - \frac{1}{x} = x - \frac{1}{x}$

$f(1) = \frac{1^2}{2} - \ln|1| = \frac{1}{2}$

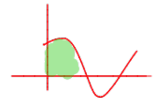
$\frac{d}{dx}(\ln|x|) = \begin{cases} \ln(x) & x>0 \\ \ln(-x) & x<0 \end{cases}$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$

$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$

Integrals

"what's inside" must match the differential



$\cos(x)$

x

function	Integral
$\int u^n \, du$ $(n \neq -1)$	$\frac{u^n}{n+1}$
$\int \frac{1}{u} \, du$	$\ln\lvert u\rvert$
$\int \cos(u) \, du$	$\sin(u)$
$\int \sin(u) \, du$	$-\cos(u)$
$\int \sec^2(u) \, du$	$\tan(u)$
$\int \sec(u)\tan(u) \, du$	$\sec(u)$
$\int e^u \, du$	e^u

$$\int \frac{1}{1+u^2} \, du = \tan^{-1}(u) \, du$$

$$\int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1}(u) \, du$$

$$\int \frac{1}{\lvert u\rvert\sqrt{u^2-1}} \, du = \sec^{-1}(u) \, du$$

match chart: Key to getting started

1. $\int \frac{(\tan^{-1} x)^3}{x^2+1}\,dx = \int \frac{u^3}{x^2+1} \cdot (1+x^2)\,du = \int u^3\,du \longrightarrow \frac{u^4}{4} + c = \frac{(\tan^{-1}x)^4}{4} + c$

identifying derivative relationships

$\int (\tan^{-1}x)^3 \cdot \frac{1}{x^2+1}\,dx$

① set $\boxed{u = \tan^{-1}x}$

② $\frac{du}{dx} = \frac{1}{1+x^2}$

③ $du = \frac{1}{1+x^2}\,dx$

goal: isolate dx

④ $\boxed{(1+x^2)\,du = dx}$

2. $\int \frac{4x}{x^2+1}\,dx = \int \frac{4x}{u} \cdot \frac{1}{2x}\,du = \int \frac{2}{u}\,du = 2\int \frac{1}{u}\,du = 2\ln|u| + c = \boxed{2\ln|x^2+1| + c}$

$\boxed{u = x^2+1}$ $\quad du = 2x\,dx$

$\frac{du}{dx} = 2x$ $\quad \boxed{\frac{1}{2x}du = dx}$

SHORTCUT recognize $u = x^2+1$, $du = 2x\,dx$, we almost have $2x\,dx$ given

$\int \frac{4x}{x^2+1}\,dx = \int \frac{4x\,dx}{x^2+1} = 2\int \frac{2x\,dx}{x^2+1} = 2\int \frac{du}{u}$

$= 2\ln|u| = 2\ln|x^2+1| + c$

3. $\int \frac{4x}{(x^2+1)^2}\,dx = \int 4x(x^2+1)^{-2}\,dx$

match: $\int u^n\,du$

$u = x^2+1$

$du = 2x\,dx$

$2\int (x^2+1)^{-2}\,2x\,dx = 2\int u^{-2}\,du = \frac{2u^{-1}}{-1} + c = \boxed{-2(x^2+1)^{-1} + c}$

4. $\int \frac{5}{x^2+1}\,dx$

① algebra

② inverse trig derivative? ✓

$5\underbrace{\int \frac{1}{x^2+1}\,dx}_{\tan^{-1}x} = 5\tan^{-1}x$

u-sub doesn't work b/c $u = x^2+1$

$\Rightarrow du = 2x\,dx$ (NO other x in integrand)

5. $\int \frac{4x+5}{x^2+1}\,dx$

Idea? $\boxed{u = x^2+1}$

$du = 2x\,dx$

$\boxed{\frac{1}{2x}du = dx}$

(there's hope this will work since $4x$ in numerator)

$= \int \frac{(4x+5)}{u}\left(\frac{1}{2x}\right)du$

$= \int \frac{2 + \frac{5}{2x}}{u}\,du$

mixed (x,u) variables $\Rightarrow$ stuck!

this is a dead end

algebra:

$\int (4x+5)(x^2+1)^{-1}\,dx = \int 4x(x^2+1)^{-1} + 5(x^2+1)^{-1}\,dx$

$= \int \underbrace{\frac{4x}{x^2+1}}_{\#2} + \underbrace{\frac{5}{x^2+1}}_{\#4}\,dx$