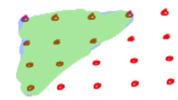
Induction & limits of partial sums

Leibniz Proved: $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \dots + \frac{1}{[n(n+1)/2]} = 2$



$\frac{1}{2}$ rectangle

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{n} + \frac{1}{n+1}$$

$$-S = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \dots - \frac{1}{n} - \frac{1}{n+1}$$

$$0 = -1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots + \frac{1}{n(n+1)} + \frac{1}{n+1}$$

$$\left(\frac{n+1}{n+1}\right)\frac{1}{n} - \frac{1}{n+1}\left(\frac{n}{n}\right) = \frac{1}{n(n+1)}$$

$$\boxed{1 - \frac{1}{n+1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots + \frac{1}{n(n+1)}}$$

Take limits:

$$\lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1$$

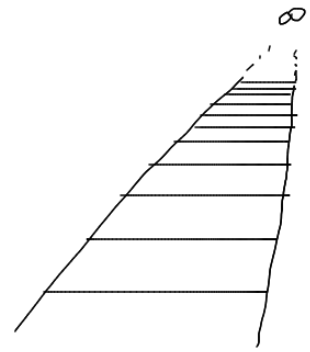
$\Rightarrow \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \dots + \frac{1}{n(n+1)}\right)$

$\to 1$ so

sum $\to 2$

Thm:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$



proof by induction:

"To climb an infinitely tall ladder you need two things"

1. get on the ladder somewhere
2. Assuming you stand anywhere on the ladder, you can climb up to the next rung.

(1) assume $n=1$ (show it's true) $\frac{1}{1(1+1)} = \frac{1}{2}$ LHS $\frac{1}{1+1} = \frac{1}{2}$ RHS

we're on the ladder

(2) Assume truth for level n. Show it's true for $n+1$

$\boxed{\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}}$. Show $\boxed{\sum_{k=1}^{n+1} \frac{1}{k(k+1)}} = \frac{n+1}{(n+1)+1}$

Same

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)[(n+1)+1]}$$

then work with it

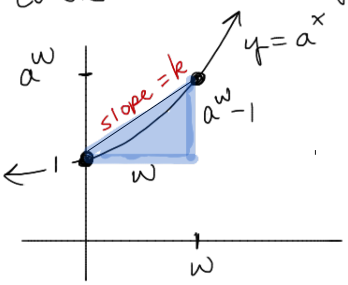
$$= \frac{(n+2)}{(n+2)}\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

[use induction hypothesis]

$\Rightarrow$ from any rung (level n) we can climb to the next.

(3) By the Principal of Mathematical Induction this statement is true for all n.

Euler's derivation of e



let $a > 1$.
Let $w =$ so small it's barely not zero.
$a^0 = 1$
when this slope is 1 ($a \approx e$)

If $k =$ slope, $\frac{a^w - 1}{w} = \frac{\text{rise}}{\text{run}} = k$ so ... $\boxed{a^w = 1 + kw}$

Idea of what's next:
If I choose any #, say 3, I can write this as:
$$3 = (1/10) \cdot 30$$
$$3 = \left(\tfrac{1}{1000}\right) \cdot 3000$$

$\forall x \in \mathbb{R}$ write $x = jw$ where $\boxed{w = \text{very small}, \ j \ \text{very large}}$

$$a^x = a^{wj} = (a^w)^j = (1 + kw)^j = \left(1 + \frac{kx}{j}\right)^j$$ Expand using binomial

$$= 1 + j\left(\frac{kx}{j}\right) + \frac{j(j-1)}{2!}\left(\frac{kx}{j}\right)^2 + \frac{j(j-1)(j-2)}{3!}\left(\frac{kx}{j}\right)^3 + \frac{j(j-1)(j-2)(j-3)}{4!}\left(\frac{kx}{j}\right)^4 + \ldots$$

$$= 1 + kx + \frac{k^2x^2}{2!} + \frac{(kx)^3}{3!} + \frac{(kx)^4}{4!} + \frac{(kx)^5}{5!} + \ldots$$

let $k = 1$; e is the $\underline{a}$ for which this true.
$$a^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

then
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$ so e is found by sub $x = 1$

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots$$

Euler's Derivation of e

(1700's) Euler approximated this constant to 23 decimal places

$$e \approx 2.71828182845904523536028$$

Euler's Identity: $e^{i\pi} = -1$

Euler's derivation of the power series for $\cos(x)$ & $\sin(x)$.

Demoire (1730) $\left(\cos z + i\sin(z)\right)^n = \cos(nz) + i\sin(nz)$ (two eq'ns)

$$\left(\cos z + i\sin(z)\right)^n = \cos(nz) + i\sin(nz)$$

$$+ \quad \left(\cos z - i\sin(z)\right)^n = \cos(nz) - i\sin(nz)$$

$$\frac{1}{2}\left[\left(\cos z + i\sin(z)\right)^n + \left(\cos z - i\sin(z)\right)^n\right] = \cos(nz)$$

$$\cos(nz) = \frac{1}{2}\Big[\cos^n(z) + n\cos^{n-1}(z)\, i\sin(z) + \frac{n(n-1)}{2!}\cos^{n-2}(z)\cdot\left(i\sin(z)\right)^2 + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}(z)\left(i\sin(z)\right)^3 + \ldots$$

$$+ \cos^n(z) - n\cdot\cos^{n-1}(z)\cdot i\sin(z) + \frac{n(n-1)}{2!}\cos^{n-2}(z)\left(i\cdot\sin(z)\right)^2 - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}(z)\left(i\cdot\sin(z)\right)^3 + \ldots$$

$$= \cos^n(z) - \frac{n(n-1)}{2!}\cos^{n-2}(z)\cdot\sin^2(z) + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}(z)\sin^4(z) + \ldots$$

let $x = nz$ & $z = $ small, $n = $ large $\Rightarrow$

$$\boxed{\begin{array}{l}\cos(z) = 1 \\ \sin(z) = z \\ n \text{ large} \Rightarrow n(n-1) \approx n^2\end{array}}$$

$$\cos(x) = 1 - \frac{n^2}{2!}\cdot z^2 + \frac{n^4}{4!} z^4 - \ldots$$