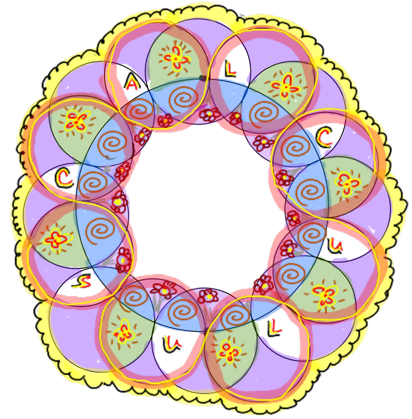
Applications Key MA161 · Exam 4

1. Evaluate the limit . . . . (a)

$$\lim_{x\to 0^+} \frac{2x^4 - 8x^2}{x^4 + 4x^2} = \frac{0}{0}$$



$$= \lim_{x\to 0} \frac{x^2(2x^2-8)}{x^2(x^2+4)} = \lim_{x\to 0^+} \frac{2x^2-8}{x^2+4}$$

$$= -2$$

(b)

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \frac{0}{0}$$

L'Hopitals

$$= \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \frac{0}{0}$$

L'H

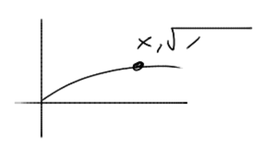
$$= \lim_{x\to 0} \frac{-\sin x}{6x} = \frac{0}{0}$$

L'H

$$= \lim_{x\to 0} \frac{-\cos x}{6} = \frac{-1}{6}$$

2. Find the point on the line $y = \sqrt{x+1}$ that is closest to the point $(8, 0)$.

Minimize distance function from $y = \sqrt{x+1}$ to $(8,0)$



$$d(x) = \sqrt{(8-x)^2 + (0-\sqrt{x+1})^2}$$

$$= \sqrt{(64 - 16x + x^2) + (x+1)}$$

$$= \sqrt{x^2 - 15x + 65}$$

$$d'(x) = \tfrac{1}{2}(x^2 - 15x + 65)^{-1/2}(2x - 15) = 0$$

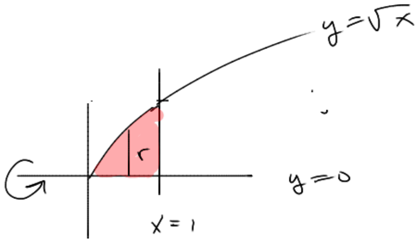
$\Rightarrow$ $2x - 15 = 0$ $x = 7.5$ so 7

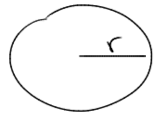
3. Consider the region bound by $x=1$, $y = 0$ and $y = \sqrt{x}$. Find the volume of the solid of revolution when the region is revolved about the:

x-axis

Int. wrt. x

Disc Method

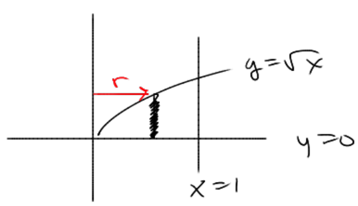




$r=\sqrt{x}$

$$\int_0^1 \pi r^2\,dx = \pi\int_0^1 x\,dx = \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

4. Revolve the region above about the **y-axis** and compute the volume of the resulting solid.



Shell Method

Integrate wrt x

$r = x$

$h = \sqrt{x}$

$$\int_0^1 2\pi x\sqrt{x}\,dx = 2\pi \int x^{3/2}\,dy$$

$$= \frac{2}{5} \cdot 2\pi x^{5/2}\Big|_0^1 = \frac{4\pi}{5}$$

5. Suppose the volume of a spherical balloon increases at a rate of $24\frac{cm^3}{sec}$. Find the rate that its diameter is increasing when the diameter is $3cm$.

Given

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} = 24 \quad \text{so} \quad \frac{dr}{dt} = \frac{24}{4\pi r^2} \quad \text{\& } r = 3 \quad \text{so} \quad \frac{dr}{dt} = \frac{24}{4\pi(3)^2}$$

$$= \frac{6}{\pi \cdot 3 \cdot 3} = \frac{2}{3\pi}\ \frac{cm}{s}$$

Want

D = diam., $D = 2r$

$$\frac{dD}{dt} = 2\frac{dr}{dt} = 2 \cdot \frac{2}{3\pi}\ \frac{cm}{s} = \frac{4}{3\pi}\ \frac{cm}{s}$$

roughly $0.5\ cm/s$

6. Find the absolute maximum and absolute minimum of the function on the indicated interval.

$$f(x) = \frac{x^4}{4} - 2x^2 + 1, \ \ [-3, 1]$$

$$f'(x) = x^3 - 4x = x(x^2 - 4) = 0$$

$\Rightarrow x = 0$

$x^2 = 4 \Rightarrow x = \pm 2$

x	-3	-2	0	1
$f(x)$	3.25	-3	1	$-\frac{3}{4}$

$20.25 = \frac{81}{4} - 18 + 1 \approx 20 - 18 + 1$
3.25

$\frac{16}{4} - 8 + 1 = -3$

Abs Max	Abs Min
$y = 3.25$ @ $x = -3$	$y = -3$ @ $x = -2$

7. A gardener is planning to build a rectangular fence which encloses 28 ft^2. One of the sides is to be made of stone which costs $10\frac{\$}{ft}$, and the remaining sides are to be made of wood which costs $4\frac{\$}{ft}$.

(a) What dimensions minimize the cost of such a fence?

w

l

Given: $A = 28$

cost of stone: \$10/ft

cost of wood: \$4/ft

Assign Variables

$w =$ width

$l =$ length

Total Cost $=$

$\$10 \cdot w + \$4l + \$4w + \$4l$

$= 14w + 8l$

(apply $A = lw = 28$, so $l = \frac{28}{w}$)

$C(w) = 14w + \frac{8 \cdot 28}{w}$

$= 14\left(w + \frac{16}{w}\right)$

(b) What is the mimimum cost?

$C(w) = 14\left(4 + \frac{16}{4}\right) = 14(8) = \$112$

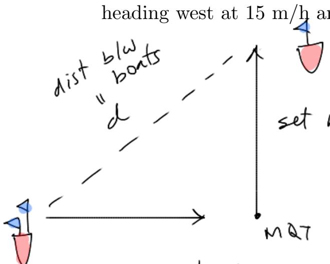
Minimize Cost:

$C'(w) = 14\left(1 - \frac{16}{w^2}\right) = 0$

$\Rightarrow 1 = \frac{16}{w^2}$ or $\boxed{w = 4}$

& $l = \frac{28}{4} = 7$

8. A boat leaves Marquette at 3:00 PM and travels due north at a speed of 10 m/h. Another boat has been heading west at 15 m/h and reaches Marquette at 5:00 PM. At what time were the boats closest together?



set $n =$ dist b/w northbound boat and MQT @ time t w/ $t =$ # hours since 3 pm

Know

$\frac{dn}{dt} = 10$ thus $n = \int \frac{dn}{dt}\,dt = 10t + C$

$n(0) = 0$

$n(0) = 10t(0) + C$

$0 \quad C = 0$

$n(t) = 10t$

set $w =$ dist of west bound boat to MQT @ time t w/ $t =$ # hours since 3 pm

Know:

$\frac{dw}{dt} = -15$ (negative, since the distance is decreasing)

$w(2) = 0$ ($t = 2$ is 5:00 pm and this boat is in MQT then)

thus

$w = \int \frac{dw}{dt}\,dt = -15t + C$

$w(2) = 0 = -15(2) + C$

so $C = 30$

$w(t) = -15t + 30$

West is $\perp$ North so it's right triangle:

$d^2 = (10t)^2 + (-15t + 30)^2$

Goal: Find t that minimizes d. Because d is positive, this is equivalent to minimizing d^2.

$(d^2)' = 2(10t) \cdot \underset{5 \cdot 2}{10} + 2(-15t + 30)(\underset{-3 \cdot 5}{-15}) = 0$ divide by 2 & 5

$= 2(10t) - 3(-15t + 30) = 0$

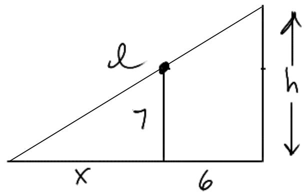
$= 20t + 45t - 90 = 0$

$65t = 90$

$t = \frac{90}{65} = \frac{18}{13} = 1\frac{6}{13} \approx 1.5$ hours

So $\approx$ 4:30

9. A fence 7 feet tall runs parallel to a tall building at a distance of 6 feet from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



Let x = dist. from bottom of ladder to building

h = height of ladder @ wall

ℓ = length of ladder

Minimize ℓ where $\ell^2 = h^2 + (x+6)^2$

since $\ell > 0$, ℓ is a minimimum whenever ℓ^2 is a min, so minimize ℓ^2.

thus $\ell^2 = \left(7+\frac{42}{x}\right)^2 + (x+6)^2$

Similar Δ's

$\frac{h}{x+6} = \frac{7}{x}$ or $h = \frac{7}{x}(x+6)$

or $h = 7 + \frac{42}{x}$

$(\ell^2)' = 2\left(7+\frac{42}{x}\right)\left(\frac{-42}{x^2}\right) + 2(x+6) = 0$

cancel the 2's and factor $\frac{7}{x}$ out of the $\left(7+\frac{42}{x}\right)$

$= \frac{7}{x}(x+6)\left(\frac{-42}{x^2}\right) + (x+6) = 0$

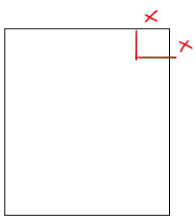
$= (x+6)\left[\left(\frac{7}{x}\right)\left(\frac{-42}{x^2}\right) + 1\right] = 0$

set both factors $= 0$ (disregard $x+6=0 \Rightarrow x=-6$)

$\frac{-7(42)}{x^3} + 1 = 0 \Rightarrow x^3 = 294$

$\boxed{x = \sqrt[3]{294}} \approx 6.6$

10. A box with an open top is to be constructed from a square piece of cardboard, 6 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.



$V = \ell \cdot w \cdot h$ $\quad w = \ell = 6-2x$ where x = length of cut

$V = x(6-2x)^2$

$V' = 1\cdot(6-2x)^2 + x\cdot 2(6-2x)(-2)$

$= (6-2x)\left[(6-2x) - 4x\right] = 0$

$= (6-2x)(6-6x) = 0$

$x = 3$ or $x = 1$

exclude (too big)

$V = 1(6-2(1))^2 = 16 \text{ ft}^3$

11. Find the equation of the tangent line to the graph of $y = (x^2 + 1)\sin x$ at $x = 0$.

slope $y' = 2x \cdot \sin x + (x^2+1)\cos x$

$y'(0) = 1$

$x_1 = 0$

$y_1 = 0$

$y = x$