GOOD LUCK ON YOUR EXAMS!

– DR. THOMPSON

Exam 4 Review

D = diameter, r = radius

1. If a snowball melts so that its surface area decreases at a rate of 1 square centimeter per minute, find the rate at which the diameter decreases when the diameter is 10 centimaters.

$\boxed{\text{Find } \frac{dD}{dt} \text{ when } d=10}$

$D = 2r \Rightarrow \frac{dD}{dt} = 2\frac{dr}{dt}$

$\frac{dS}{dt} = -1$ & $S = 4\pi r^2 \Rightarrow \frac{ds}{dt} = 8\pi r \cdot \frac{dr}{dt}$

$\Rightarrow -1 = 8\pi r \cdot \frac{dr}{dt}$

$\Rightarrow \frac{-1}{8\pi r} = \frac{dr}{dt}$

& when $D = 10$. then $r = 5$ so

$\frac{dr}{dt} = \frac{1}{8\pi(5)} = \frac{-1}{40\pi} \frac{cm}{s}$

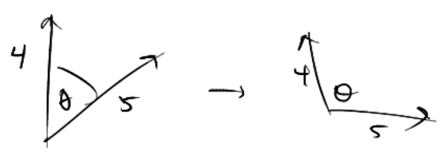
so $\frac{dD}{dt} = 2\left(\frac{-1}{40\pi}\frac{cm}{s}\right) = \boxed{-.0159 \frac{cm}{s}}$

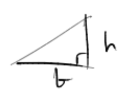
The diameter is decreasing at a rate of .0159 cm per second.

2. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 radians per second. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.

— Area of triangle —

Area = $\frac{1}{2} bh \sin\theta$





$h \rightarrow \frac{1}{2} b \cdot h \cdot \sin 90$

$A = \frac{5 \cdot 4}{2} \sin\theta$

$\Rightarrow \frac{dA}{dt} = 10\cos\theta \cdot \frac{d\theta}{dt}$ & given info $\Rightarrow 0.06 = \frac{d\theta}{dt}$

$\Rightarrow \theta = \frac{\pi}{3}$

$= 10\cos\left(\frac{\pi}{3}\right) \cdot 0.06 = \boxed{.3 \frac{m}{s}}$

The area is increasing at a rate of .3 meter per second.

3. Find an equation of the tangent line to the function at the given point.
(a) $f(x) = e^{x/10}, x = 0$
(b) $f(x) = \sqrt{x}, x = 16$

(a) $f'(x) = e^{x/10} \cdot (1/10) \Rightarrow f'(0) = e^0 \cdot (1/10) = 1/10 =$ slope

$x = 0 \Rightarrow y = f(0) = e^{0/10} = 1$ so the point is $(0,1)$

$y - 1 = 1/10 (x - 0) \rightarrow \boxed{y = 1/10 \cdot x + 1}$

(b) $f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$ so $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{2 \cdot 4} = \frac{1}{8}$

$x = 16 \Rightarrow y = f(16) = \sqrt{16} = 4 \Rightarrow (16, 4)$

$y - 4 = \frac{1}{8}(x - 16)$ $\boxed{y = \frac{1}{8} x + 2}$

4. Use your work in #3 find estimates . . ..
(a) $e^{0.1/10}$, $e^{.25/10}$
(b) $\sqrt{16.5}$, $\sqrt{17.5}$

(a) $e^{.1/10} \approx \frac{1}{10}(.1) + 1 = 1.01$

$e^{.25/10} \approx \frac{1}{10}(.25) + 1 = 1.025$

(b) $\sqrt{16.5} \approx \frac{1}{8}(16.5) + 2 = 4.0625$

$\sqrt{17.5} \approx \frac{1}{8}(17.5) + 2 = 4.1875$

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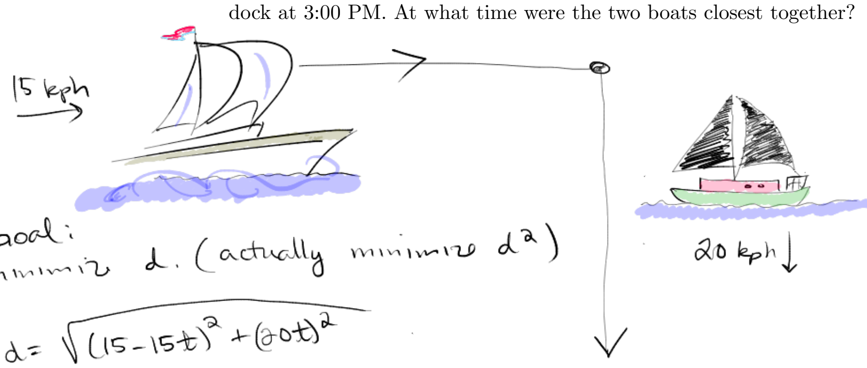
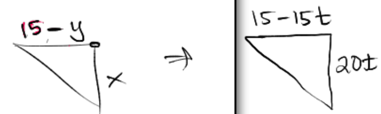
Given:
The second boat covers 15 k in one hour, thus at 2 pm was 15 k from dock

x = dist to dock of boat #1
y = dist covered since 2:00pm by boat #2

5. A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 kph. Another boat has been heading due east at 15 kph and reaches the same dock at 3:00 PM. At what time were the two boats closest together?





$\frac{dx}{dt} = 20 \Rightarrow x = 20t$

$\frac{dy}{dt} = 15 \Rightarrow y = 15t$

Goal: minimize d. (actually minimize d^2)

$$d = \sqrt{(15-15t)^2 + (20t)^2}$$

$$D = d^2 = (15-15t)^2 + (20t)^2$$

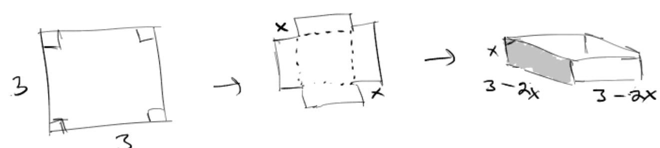
$$D' = \cancel{2}(15-15t)(\cancel{-15}^{\,-3\cdot 5}) + \cancel{2}(20t)\cdot\cancel{20}^{\,4\cdot 5} = 0$$

$$\Rightarrow (-3)(\cancel{15}_{5\cdot 3})(1-t) + \cancel{80t}_{5\cdot 16} = (-9)(1-t) + 16t = 0$$

$$= -9 + 25t = 0 \Rightarrow t = \frac{9}{25} \Longrightarrow \frac{9}{\cancel{25}_{5}} \times \frac{\cancel{60}^{12}\text{ min}}{1\text{ hr}} = \frac{108}{5} = 21.6 \text{ min past 2pm}$$

$\Rightarrow \boxed{2{:}21\text{ pm}}$

6. A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.



$$V = x(3-2x)^2$$

$$V' = (3-2x)^2 + x(2(3-2x)(-2))$$

$$= (3-2x)((3-2x) - 4x)$$

$$= (3-2x)(3-6x) = 0$$

$$3 = 2x \quad , \quad 3 = 6x$$

$$\frac{3}{2} = x \quad , \quad \frac{1}{2} = x$$

The largest volume is

$$V\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)\left(3 - 2\left(\tfrac{1}{2}\right)\right)^2 = \left(\tfrac{1}{2}\right)(2)^2 = 2 \text{ ft}^3$$

Second Derivative Test:

$$V'' = (-2)(3-6x) + (3-2x)(-6)$$

$$= 12x - 6 + 12x - 18$$

$$= 24x - 24 = 24(x-1)$$

$$V''\left(\tfrac{3}{2}\right) = 24\left(\tfrac{3}{2} - 1\right) > 0$$

$$V''\left(\tfrac{1}{2}\right) = 24\left(\tfrac{1}{2} - 1\right) < 0 \Rightarrow$$

Volume has a local maximum at $x = \frac{1}{2}$.

7. Evaluate the limit . . . .

(a)

$$\lim_{x\to+\infty}\frac{(\ln x)^2}{x} = \frac{\infty^2}{\infty} = \frac{\infty}{\infty} \quad \text{L'H}$$

$$= \lim_{x\to+\infty} \frac{2(\ln x)\frac{1}{x}}{1} = \lim_{x\to+\infty}\frac{2\ln x}{x} = \lim_{x\to+\infty}\frac{2\left(\frac{1}{x}\right)}{1} = \lim_{x\to+\infty}\frac{2}{x} = \boxed{0}$$

Makes sense b/c logarithmic growth is slower than linear growth.

(b)

$$\lim_{x\to0}\frac{\sin x - x}{x^3} = \frac{0}{0} \quad \text{L'H} = \lim_{x\to0}\frac{\cos x - 1}{3x^2} = \frac{0}{0}$$

$$= \lim_{x\to0}\frac{-\sin x}{6x} = \frac{0}{0}$$

$$= \lim_{x\to0}\frac{-\cos x}{6} = \boxed{-\frac{1}{6}}$$

(c)

$$\lim_{x\to0}\frac{x+\sin x}{x+\cos x}$$

plug in

$$\frac{0+\sin(0)}{0+\cos(0)} = \frac{0}{1} = \boxed{0}$$

(d)

$$\lim_{x\to0}\left(1-\frac{x}{2}\right)^{1/x}$$

$y = \left(1-\frac{x}{2}\right)^{1/x}$

want $\lim_{x\to0} y$. Take ln both sides

$$\ln y = \ln\left(1-\frac{x}{2}\right)^{1/x} = \frac{1}{x}\ln\left(1-\frac{x}{2}\right) = \frac{\ln\left(1-\frac{x}{2}\right)}{x}$$

$$\lim_{x\to0}\ln y \overset{\text{L'H}}{=} \frac{\lim_{x\to0}\left[\frac{\left(-\frac{1}{2}\right)}{\left(1-\frac{x}{2}\right)}\right]}{1} = \lim_{x\to0}\frac{-1}{2\left(1-\frac{x}{2}\right)} = \frac{-1}{2}$$

$\Rightarrow$ since

$\lim_{x\to0}\ln y = \ln\lim_{x\to0} y$

take e^x both sides

$$e^{\ln\lim_{x\to0} y} = e^{-1/2}$$

$$\boxed{\lim_{x\to0} y = e^{-1/2}}$$

8. Find the absolute maximum and absolute minimum of the function on the indicated interval.

(a) $f(x) = 3x^4 + 8x^3 - 18x^2 + 5$, $[-4, 2]$

(b) $f(x) = 3x^4 - 4x^3 - 12x^2$, $[-3, 1]$

(a) $f'(x) = 12x^3 + 24x^2 - 36x = 0$

$= 12x(x^2 + 2x - 3) = 0$

$= 12x(x+3)(x-1) = 0$

$\Rightarrow x = 0, 1, -3$

x	0	1	-3	-4	2
$f(x)$	5	-2	-130	-27	45

Abs. Max: $f(x) = 45$ at $x = 2$

Abs Min: $f(x) = -130$ at $x = -3$

(b) $f'(x) = 12x^3 - 12x^2 - 24x = 0$

$= 12x(x^2 - x - 2) = 0$

$= 12x(x-2)(x+1) = 0$

$x = 0, -1, 2$ but 2 is outside $[-3,1]$

x	0	-1	-3	1
$f(x)$	0	-5	243	-13

Abs Max: $f(x) = 243$ at $x = -3$

Abs Min: $f(x) = -13$ @ $x = 1$

9. Find the average value of the function $\sin x$ on the interval $[0, \pi]$.

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin x = -\frac{1}{\pi}\cos(x)\Big|_0^{\pi} = -\frac{1}{\pi}\left(\cos\pi - \cos 0\right)$$

$$= -\frac{1}{\pi}(-1-1) = \boxed{\frac{2}{\pi}}$$

10. Find the average value of the function $\frac{1}{\sqrt{1-x^2}}$ on the interval $[0, 0.5]$.

$$\frac{1}{.5-0}\int_0^{.5} \frac{1}{\sqrt{1-x^2}}\,dx = 2\cdot\sin^{-1}(x)\Big|_0^{.5} = 2\cdot\left(\sin^{-1}(.5) - \sin^{-1}(0)\right)$$

$$= 2\left(\frac{\pi}{6} - 0\right) = \boxed{\frac{\pi}{3}}$$

11. Sketch the region enclosed by the given curves. Use an integral to find the area enclosed.

(a) $y = x$, $y = x^2$

(b) $y = x^2 - 2x$, $y = x + 4$ see following pages

12. Find the volume when the area enclosed in #11(a) is rotated . . .

(a) around the x-axis

(b) around the line $x = -1$

(c) around the y-axis

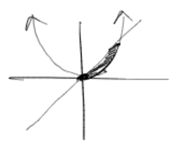
(d) around the line $y = 2$ see following pages

13. When a particle is located a distance x meters from the origin, a force of $\frac{1}{1+x^2}$ newtons acts on it. How much work is done in moving the particle from $x = 0$ to $x = 1$?

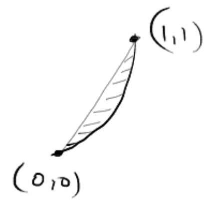
$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}\text{ J}}$$

#11 $y = x$, $y = x^2$



$\longrightarrow$



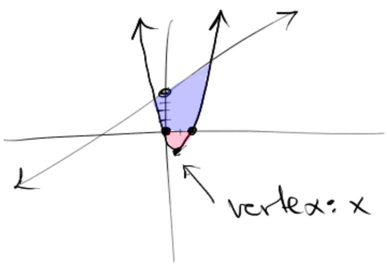
(a)

Area

$$= \int_0^1 x - x^2 \, dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \boxed{\frac{1}{6}}$$

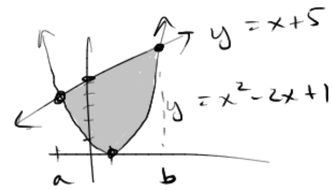
(b)

$y = x^2 - 2x$ $\qquad$ $y = x + 4$



translate up by 1.

$\longrightarrow$



vertex: $x \; -\frac{b}{2a} = \frac{-(-2)}{2(1)} = 1$, & $y\big|_{x=1} = 1-2 = -1$

$\Rightarrow$ vertex $(1,-1)$

Find bounds: $x^2 - 2x + 1 = x + 5$

$$x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

$$\boxed{\begin{array}{l} x = 4 \\ x = -1 \end{array}}$$

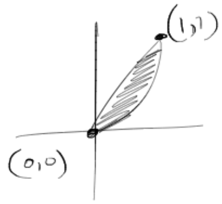
$$\int_{-1}^{4} (x+5) - (x^2 - 2x + 1) \, dx$$

$$= \int_{-1}^{4} -x^2 + 3x + 4 \, dx$$

$$= -\frac{x^3}{3} + \frac{3x^2}{2} + 4x \Big|_{-1}^{4}$$

$$= -\frac{64}{3} + 3 \cdot 8 + 16 - \left(\frac{1}{3} + \frac{3}{2} - 4\right) = 20.83$$

11 (a) about x-axis



vertical slice

$r_2 = x^2$ $\quad r_1 = x$

$$V = \int_0^1 \pi x^2 - \pi x^4 \, dx$$

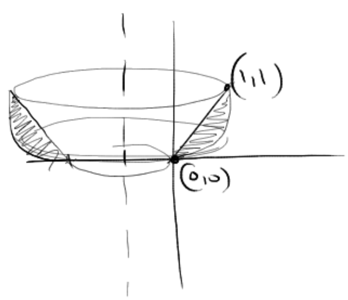
$$= \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1$$

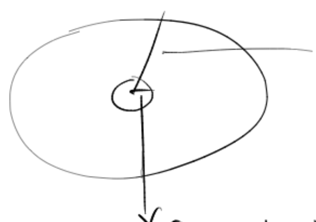
$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \pi\left(\frac{5-3}{15}\right) = \boxed{\frac{2\pi}{15}}$$

$$A = \pi r_1^2 - \pi r_2^2$$

$$= \pi x^2 - \pi x^4$$

$x = -1$ Integrate wrt y. $\Rightarrow$ $x = y$, $x = \sqrt{y}$





$r_1 = 1 + \sqrt{y}$

$r_2 = 1 + y$

$$\Rightarrow A = \pi(1+\sqrt{y})^2 - \pi(1+y)^2$$

$$= \pi\left((1 + 2\sqrt{y} + y) - (1 + 2y + y^2)\right)$$

$$= \pi\left(-y^2 - y + 2\sqrt{y}\right)$$

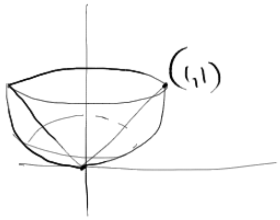
$$V = \pi\int_0^1 (-y^2 - y + 2\sqrt{y})\,dy = \pi\left(-\frac{y^3}{3} - \frac{y^2}{2} + \frac{2y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right)\Big|_0^1 = \pi\left(-\frac{1}{3} - \frac{1}{2} + \frac{4}{3}\right)$$

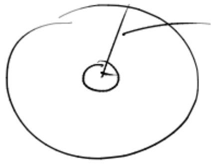
$$= \pi\left(-\frac{2}{6} - \frac{3}{6} + \frac{8}{6}\right) = \frac{\pi}{2}$$

y-axis Horizontal Slice → Integrate wrt y. ⇒ change into functions of y

$y = x \Rightarrow x = y$

$y = x^2 \Rightarrow x = \sqrt{y}$





$r_1 = \sqrt{y}$

$r_2 = y$

$A = \pi(\sqrt{y})^2 - \pi(y)^2$

$= \pi(y - y^2)$

$V = \pi\int_0^1 y - y^2\,dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$

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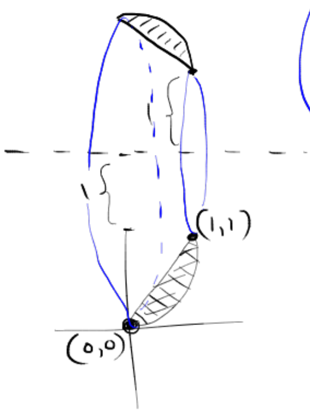
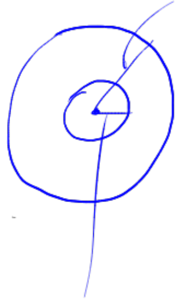
about $y = 2$





$A = \pi(2-x^2)^2 - \pi(2-x)^2$

$= \pi\left((4-4x^2+x^4) - (4-4x+x^2)\right)$

$= \pi\left(x^4 - 5x^2 + 4x\right)$

$V = \int_0^1 \pi\left(x^4 - 5x^2 + 4x\right)dx$

$= \pi\left(\frac{x^5}{5} - \frac{5x^3}{3} + \frac{4x^2}{2}\right)\Big|_0^1$

$= \pi\left(\frac{1}{5} - \frac{5}{3} + 2\right) = \pi\left(\frac{3 - 25 + 30}{15}\right)$

$= \frac{8\pi}{15}$