MA161 Final Exam Guide

1. Limits

(a) $\lim_{x\to 3}\left[\frac{1}{(x-3)} - \frac{1}{(x^2-5x+6)}\right] = \frac{x-2-1}{x^2-5x+6} = \frac{x-3}{(x-2)(x-3)} = \frac{1}{x-2} = 1$

divide top & bottom by x

(b) $\lim_{x\to+\infty}\left[\sqrt{x^2+8x} - x\right] \cdot \frac{\sqrt{x^2+8x}+x}{\sqrt{x^2+8x}+x} = \lim \frac{x^2+8x-x^2}{\sqrt{x^2+8x}+x} = \lim_{x\to+\infty} \frac{8x}{\sqrt{x^2+8x}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim_{x\to+\infty} \frac{8}{\frac{\sqrt{x^2+8x}}{x}+1} = \lim_{x\to+\infty} \frac{8}{\frac{\sqrt{x^2+8x}}{\sqrt{x^2}}+1} = \lim_{x\to+\infty} \frac{8}{\sqrt{\frac{x^2+8x}{x^2}}+1} = \lim_{x\to+\infty} \frac{8}{\sqrt{1+\frac{8}{x}}+1} = \frac{8}{\sqrt{1}+1}$

$= \frac{8}{2}$

$= 4$

(c) $\lim_{x\to-4^+} \frac{1}{x+4} = +\infty$

(d) $\lim_{x\to-4} \frac{1}{x+4} =$ DNE

(e) $\lim_{x\to-4} \frac{-7}{(x+4)^2} = -\infty$

(f) $\lim_{x\to+\infty} e^x \sin x =$ DNE, but why???

(g) $\lim_{x\to+\infty} \frac{\sin x}{x} = 0$, (use squeeze thm)

(h) $\lim_{x\to 0} \frac{\sin x}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$

(i) $\lim\limits_{x\to\pi/2} \dfrac{\sin x}{x} =$ $\dfrac{2}{\pi}$

(j) $\lim\limits_{x\to 1} \dfrac{x^7-1}{x^5-1} \overset{L'H}{=} \lim\limits_{x\to 1} \dfrac{7x^6}{5x^4} = \dfrac{7}{5}x^2 = \dfrac{7}{5}$

(k) $\lim\limits_{x\to 0} \left(1+\dfrac{1}{2}x\right)^{1/x} = y$ so $\boxed{e^{\frac{1}{2}} = y}$

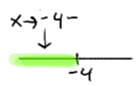
$\ln \lim\limits_{x\to 0} (1+\tfrac{1}{2}x)^{1/x} = \ln y$

$= \lim \ln\left((1+\tfrac{1}{2}x)^{1/x}\right) = \lim\limits_{x\to 0} \dfrac{1}{x}\ln(1+\tfrac{1}{2}x) = \lim\limits_{x\to 0} \dfrac{\ln(1+\frac{1}{2}x)}{x} = \dfrac{0}{0}$

L'H → $\lim\limits_{x\to 0} \dfrac{\frac{\frac{1}{2}}{1+\frac{1}{2}x}}{1} = \dfrac{1}{2}$

(l) $\lim\limits_{x\to 2} 7 =$ 7

(m) $\lim\limits_{x\to 2} x^3+3 =$ 11



(n) $\lim\limits_{x\to -4^-} \dfrac{1}{x+4} =$ Note: $x\to -4^-$ means $x \approx -4$ but $x<4$, so $x \approx -5, -4.5, -4.1, -4.01, \ldots$

x	-5	-4.5	-4.1	-4.01
$\frac{1}{x+4}$	$\frac{1}{-1}$	$\frac{1}{-1/2}$	$\frac{1}{-1/10} = -10$	$\frac{1}{-\frac{1}{100}} = -100$

$\Rightarrow -\infty$

2. Use the limit definition of the derivative to find $f'(x)$.

$$f(x) = \frac{1}{2x+1}$$

$\lim\limits_{h\to 0} \dfrac{\dfrac{1}{2(x+h)+1} - \dfrac{1}{2x+1}}{h} = \lim\limits_{h\to 0} \dfrac{2x+1-[2(x+h)+1]}{[2(x+h)+1][2x+1]} \cdot \dfrac{1}{h} = \lim\limits_{h\to 0} \dfrac{2x+1-2x-2h-1}{\text{stuff}} \cdot \dfrac{1}{h}$

$= \lim \dfrac{-2h}{\text{stuff}} \cdot \dfrac{1}{h} = \lim\limits_{h\to 0} \dfrac{-2}{\text{stuff}}$

$\lim\limits_{h\to 0} \dfrac{-2}{[2(x+h)+1][2x+1]} = \dfrac{-2}{(2x+1)^2}$

3. Find $f'(x)$

(a) $f(x) = 2$

0

(b) $f(x) = x$

1

(c) $f(x) = e^{2x} - 3x$

$2e^{2x} - 3$

(d) $f(x) = \dfrac{1}{1+x^2}$

$-(1+x^2)^{-2} \cdot 2x$

(e) $f(x) = \tan^{-1} x$

$\dfrac{1}{1+x^2}$

(f) $f(x) = \sqrt{x^4 + 5}$

$\frac{1}{2}(x^4+5)^{-\frac{1}{2}} \cdot 4x^3$

(g) $f(x) = \sin^3 x$

$3\sin^2 x \cdot \cos x$

(h) $f(x) = x^3 e^{2x+5}$

$3x^2 \cdot e^{2x+5} + x^3\, 2e^{2x+5}$

(i) $f(x) = x^x$ $= y$

$\ln y = \ln x^x = x \ln x$

$\frac{y'}{y} = \ln x + \frac{x}{x} = \ln x + 1$ so $y' = [\ln x + 1] \cdot x^x$

4. Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 3x + 1$ on the interval [0,3]. $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 0$; $x = \pm 1$

x	f(x)
-1	3
0	1
1	-1
3	19

AbsMin: 1

AbsMax: 19

5. Find an equation of the tangent line to the graph of $y = x^5$ at $x = 2$. Then use it to approximate $(2.04)^5$.

$y'(2) = 5(2)^4 = 80$

$y\big|_{x=2} = 32$

$y - 32 = 80(x-2)$

$y = 80x - 128$

$y\big|_{x=2.04} \approx 80(2.04) - 128$

$= 163.2 - 128 = 35.2$

6. (a) Find the point on the line $y = x + 1$ that is closest to the point (1,1). Hint, use the distance formula between (1,1) and (x,x+1).

   (b) If 1200 square centimeters of material is available to make an open box with a square base, find the largest possible volume for the box.

(a) $d=\sqrt{(x-1)^2+(x+1-1)^2}$

$=\sqrt{2x^2-2x+1}$

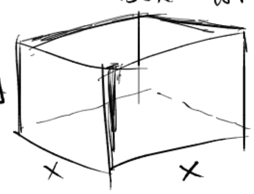
$d'=0 \iff 4x-2=0$

$x=1/2$

so $y=3/2$

$\left(\frac{1}{2},\frac{3}{2}\right)$

(b) Goal: produce largest volume with a box whose surface area is 1200 cm².



$SA = x^2+4xy = 1200$ / so... $y=\frac{1200-x^2}{4x}$

$y=\frac{300}{x}-\frac{1}{4}x$

$V = y\cdot x^2$

$=\left[\frac{300}{x}-\frac{1}{4}x\right]x^2 = 300x-\frac{1}{4}x^3$

$\frac{dV}{dx} = -\frac{3}{4}x^2+300 = 0$

$x^2 = 300\cdot\frac{4}{3} = 400$

$x=20$

$V=\left[\frac{300}{20}-\frac{1}{4}(20)\right]\cdot\left[20^2\right] = [15-5]400 = 4000 \text{ cm}^3$

7. (a) If the radius of a circle is increasing at a rate of 1.5 cm/s, find the rate at which the area is changing when the radius is $4cm$.

   (b) Ship A is 150 miles west of Ship B. Ship A sails south at a rate of 30 miles per hour. Ship B sails north at a rate of 40 miles an hour. Find the rate at which the distance between the ships is changing two hours later.

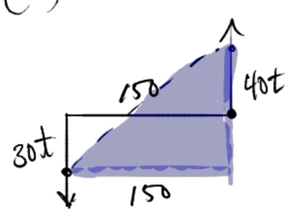
(a) $\frac{dr}{dt}=1.5$ (given)

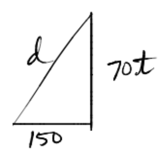
$A=\pi r^2$ so

$\frac{dA}{dt}=2\pi r\cdot\frac{dr}{dt}$

$2\pi(4)\cdot(1.5) = 12\pi$

(b)



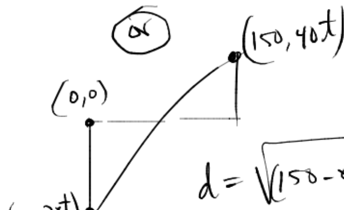


$D=\sqrt{70^2t^2+150^2}$

$\frac{dD}{dt}=\frac{1}{2}\left[70^2t^2+150^2\right]^{-\frac{1}{2}}\cdot\left[4900\cdot 2t\right]$

$\left.\frac{dD}{dt}\right|_{t=2} = \frac{4900\cdot 2}{\sqrt{70^2\cdot 4+150^2}} = .43 \frac{\text{miles}}{\text{hour}}$

or



$d=\sqrt{(150-0)^2+(40t-(-30t))^2} = \sqrt{150^2+(70t)^2}$

same!

8. Evaluate the indefinite integral.

(a) $\int \sin^2(x)\cos(x)dx =$ $\frac{\sin^3}{3} + C$

(b) $\int e^{3x-2}dx =$ $\frac{1}{3}e^{3x-2} + C$

(c) $\int \frac{2x}{x^2+1}dx =$

$\ln|x^2+1| + C$

(d) $\int \frac{7+2x}{x^2+1}dx =$

$7\tan^{-1}x + \ln|x^2+1| + C$

(e) $\int 4x^3(x^4+1)^5dx =$

$\frac{(x^4+1)^6}{6}$

(f) $\frac{1}{2}\int \frac{2x^3 \cdot 2}{\sqrt{x^4+5}}dx =$ $= \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{\frac{1}{2}u^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{u} + C$

$u = x^4+5$

$du = 4x^3$

$= \sqrt{x^4+5} + C$

9. Find the area of the region that lies under the graph of

$$f(x) = \sqrt[4]{x}$$

between $x = 1$ and $x = 16$.

$$\int_1^{16} x^{1/4}\,dx = \frac{4}{5}x^{5/4}\Big|_1^{16} = \frac{4}{5}x\cdot x^{1/4}\Big|_1^{16} = \frac{4}{5}[16\cdot 2 - 1]$$

$$= \frac{4}{5}[31] = \frac{124}{5} = 20.8$$

10. Given the following information, find $f(x)$.

   (a) $f''(x) = 6x - 4$
   (b) $f'(-1) = 13$
   (c) $f(2) = 20$

$f' = \frac{6x^2}{2} - 4x + c$

$\underset{13}{\overset{\shortparallel}{f'(-1)}} = 3 + 4 + c$ $\quad \boxed{c = 6}$

$f'(x) = 3x^2 - 4x + 6$

$f(x) = \int f'(x)\,dx = x^3 - 2x^2 + 6x + c$

$\underset{20}{\overset{\shortparallel}{f(2)}} = 8 - 8 + 12 + c$

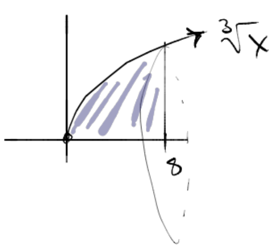
$c = 8$

$f(x) = x^3 - 2x^2 + 6x + 8$

11. Find the volume of the solid obtained by revolving the region between the x-axis and the curve

$$y = \sqrt[3]{x}$$

over $0 \le x \le 8$, around the x-axis.



Disc: $r = \sqrt[3]{x}$

$A = \pi r^2$

$= \pi\sqrt[3]{x^2}$

$$V = \int_0^8 \pi\sqrt[3]{x^2}\,dx = \pi\frac{x^{5/3}}{5/3}\Big|_0^8$$

$$\frac{3\cdot 32\pi}{5} = \frac{96\pi}{5}$$