Math 161 - Calculus - Exam 1 - Guide **Name:** Solutions

September 12, 2024

On the exam you must show your work to receive full credit.

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1. Evaluate the following limits.:

(1.1) $\lim_{x\to 4}\frac{1}{x} = \frac{1}{4}$

(1.2) $\lim_{x\to 5} 3 = 3$

(1.3) $\lim_{x\to 4}\frac{1}{x-4}$ DNE b/c $\lim_{x\to 4^-}\frac{1}{x-4} = -\infty$ and $\lim_{x\to 4^+}\frac{1}{x-4} = +\infty$

(1.4) $\lim_{x\to 5}\frac{-1}{(x-5)^2} = -\infty$ b/c $\frac{-1}{(4.99-5)^2} \approx \frac{-1}{(5.01-5)^2}$

(1.5) $\lim_{x\to 0}\frac{\sqrt{x^3+13}\sin(x)}{x} = \lim_{x\to 0}\sqrt{x^3+13}\cdot\underbrace{\lim_{x\to 0}\frac{\sin(x)}{x}}_{=1 \text{ (from class)}} = \lim_{x\to 0}\sqrt{x^3+13} = \sqrt{13}$

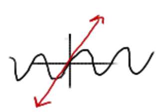
(1.6) $\lim_{x\to +\infty}\frac{1}{x-4} \approx \frac{1}{\text{large positive}} = 0$

(1.7) $\lim_{x\to +\infty}\frac{\cos(2x)}{x} = 0$ by Squeeze theorem: $\Rightarrow$ $-1 \le \cos(2x) \le 1$

$-\frac{1}{x} \le \frac{\cos(2x)}{x} \le \frac{1}{x}$

$\to 0$ $\quad$ $\to 0$

(1.8) $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$



(1.9) $\lim_{x\to +\infty} e^x\cos(x)$ DNE

(1.10) $\lim_{x\to 4}\left[\frac{2}{x-4} - \frac{2}{x^2-7x+12}\right] = \lim \frac{2(x-3)}{(x-4)(x-3)} - \frac{2}{(x-4)(x-3)} = \lim \frac{\overbrace{2x-8}^{2(x-4)}}{(x-4)(x-3)}$

$= \lim_{x\to 4}\frac{2}{x-3} = 2$

(1.11) $\lim_{x\to 0}\frac{\sin(7x)}{\sin(5x)}$

$\| $

$\lim \frac{\sin(7x)\cdot\frac{1}{x}}{\sin(5x)\cdot\frac{1}{x}} = \lim \frac{\frac{\sin(7x)}{x}}{\frac{\sin(5x)}{x}} = \lim \frac{\frac{\sin(7x)}{x}\cdot\frac{7}{7}}{\frac{\sin(5x)}{x}\cdot\frac{5}{5}} = \lim \frac{7\cdot\frac{\sin(7x)}{7x}}{5\cdot\frac{\sin(5x)}{5x}}$

$= \frac{7}{5}\cdot\frac{\lim_{x\to 0}\frac{\sin 7x}{7x}}{\lim_{x\to 0}\frac{\sin(5x)}{5x}} = \frac{7}{5}\cdot\frac{1}{1} = \frac{7}{5}$

2. (Give a short written response) What does the derivative tell you about a function?

① Slope of the tangent line to the graph.

② Instantaneous rate of change.

3. Use the definition of the derivative to compute $f'(x)$.

(3.1) $f(x) = \dfrac{3}{x-1}$

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{3}{x+h-1} - \dfrac{3}{x-1}}{h} = \lim \frac{\dfrac{3(x-1) - 3(x+h-1)}{(x+h-1)(x-1)}}{h} = \lim \frac{3x-3-3x-3h+3}{(x+h-1)(x-1)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-3h}{(x+h-1)(x-1)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-3}{(x+h-1)(x-1)} = \frac{-3}{(x-1)^2}$$

(3.2) $f(x) = 5\sqrt{x+2}$

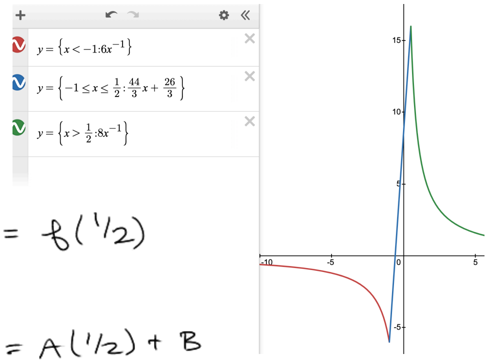
$$f'(x) = \lim_{h\to 0} \frac{5\sqrt{(x+h)+2} - 5\sqrt{x+2}}{h} = 5\cdot\lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}$$

$$= 5\cdot\lim_{h\to 0} \frac{(x+h+2) - (x+2)}{h\cdot(\sqrt{x+h+2} + \sqrt{x+2})} = 5\cdot\lim_{h\to 0} \frac{h}{h\cdot(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= 5\cdot\lim_{h\to 0} \frac{1}{1\cdot(\sqrt{x+h+2} + \sqrt{x+2})} = \frac{1}{2\sqrt{x+2}}$$

4. For what values of A and B is $f(x)$ continuous?

$$f(x) = \begin{cases} 6x^{-1} & x < -1 \\ Ax + B & -1 \leq x \leq \frac{1}{2} \\ 8x^{-1} & x > \frac{1}{2} \end{cases}$$



1

Need: $\underbrace{\lim_{x \to -1^-} f(x)} = f(-1)$

So $6(-1)^{-1} = A(-1) + B$

$-6 = \frac{6}{-1}$; $-A + B$

$\Rightarrow \boxed{-6 = B - A}$

2

Need $\underbrace{\lim_{x \to 1/2} f(x)} = f(1/2)$

So $8(1/2)^{-1} = A(1/2) + B$

$16 = 8 \cdot 2 = \frac{A}{2} + B$

$\boxed{16 = A/2 + B}$

3 Combine:

Since $B = A - 6$ we get $16 = A/2 + A - 6$

$22 = \frac{3A}{2}$

$44 = 3A$

$A = 44/3$

so

$B = 44/3 - 6$

$= 44/3 - 18/3 = 26/3$

5. For functions, what is the relationship between the concepts: derivative, increasing and decreasing?

Increasing functions: their derivative is positive

decreasing functions: their derivative is negative

6. Find all solutions

(6.1) $3e^x + 5 = e^x + 11$

$2e^x = 6$

$e^x = 3$

$\ln(e^x) = \ln 3$

$\boxed{x = \ln 3}$

(6.2) $\left(1 + \frac{0.06}{12}\right)^{2x} = 4$

$\ln\left(1 + \frac{0.06}{12}\right)^{2x} = \ln 4$

$2x \cdot \ln\left(1 + \frac{0.06}{12}\right) = \ln 4$

$\boxed{x = \frac{\ln 4}{2 \cdot \ln\left(1 + \frac{0.06}{12}\right)}}$

or $\left(1 + \frac{0.06}{12}\right)^{2x} = 4$

$\Rightarrow \left[\left(1 + \frac{0.06}{12}\right)^{2x}\right]^{1/2} = 4^{1/2}$

$\left(1 + \frac{0.06}{12}\right)^{x} = 2$

$\log_2\left(1 + \frac{0.06}{12}\right)^{x} = \log_2 2$

$x \cdot \log_2\left(1 + \frac{0.06}{12}\right) = 1$

$\boxed{x = \frac{1}{\log_2\left(1 + \frac{0.06}{12}\right)}}$

(6.3) $\frac{50}{1 + 2e^{3x}} = 10$

$50 = 10(1 + 2e^{3x})$

$5 = (1 + 2e^{3x}) = 1 + 2e^{3x}$

$4 = 2e^{3x}$

$2 = e^{3x}$

$\ln(2) = \ln(e^{3x}) = 3x$

$\frac{\ln(2)}{3} = x$

7. Find $f^{-1}(x)$.

(7.1) $f(x) = \frac{1 - 4x}{3x + 2}$

① set $y = \frac{1 - 4x}{3x + 2}$

② swap $x = \frac{1 - 4y}{3y + 2}$

③ Solve

$x(3y + 2) = 1 - 4y$

$3xy + 2x = 1 - 4y$

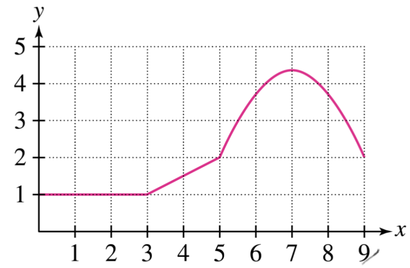
$3xy + 4y = 1 - 2x$

$y(3x + 4) = 1 - 2x$

$y = \frac{1 - 2x}{3x + 4}$

$\boxed{f^{-1}(x) = \frac{1 - 2x}{3x + 4}}$

8. Assume the graph of $f(x)$ is below. For what values of a is $f'(a)$ positive, negative and zero?



positive	negative	zero
(3,5) (5,7)	(7,9)	(0,3)

Important:

$f'(3)$ and $f'(5)$ do not exist so 5 and 7 <u>don't</u> appear

$$f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \begin{cases} \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = 0 \\ \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \frac{1}{2} \end{cases} \Rightarrow \text{thus } f'(3) \text{ DNE}$$

9. Suppose that f has a domain of [7, 17] and a range of [2, 17].

(9.1) What are the domain and range of the function y = f(x) + 4?

(9.2) What are the domain and range of the function y = f(x + 4)?

(9.1) $y = f(x) + 4$

(vertical shift) $\Rightarrow$ no affect on domain, but pushes range up by 4

Domain: [7, 17]

range: [6, 21]

(9.2) $f = f(x+4)$

horizontal shift $\Rightarrow$ shifts domain to left, keeps range the same

Domain: since domain of $f(x)$ is [7,17] we must have:

$x+4 \in [7,17]$

so

$x \in [3,13]$

range: [2,17]

10. According to Peak Oil Theory, first proposed in 1956, the total amount of crude oil $Q(t)$ produced worldwide up to time t has a graph like the one shown below.

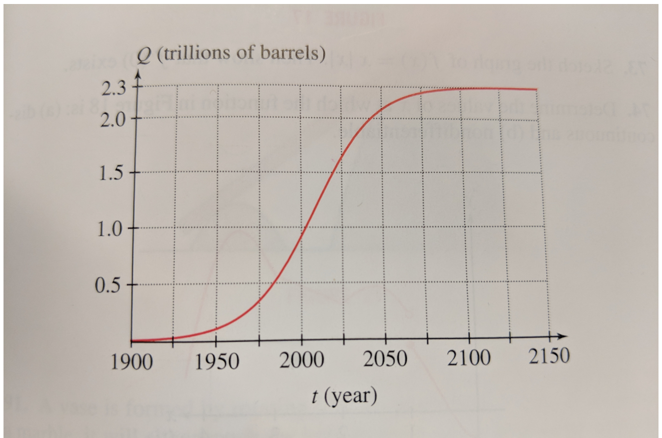


FIGURE 14 Total oil production up to time t.

(10.1) $\dfrac{Q(2020) - Q(1900)}{2020 - 1900}$

$\approx \dfrac{1.5}{120} \dfrac{\text{trillion barrels}}{\text{year}}$

$\approx \dfrac{1500}{120} \dfrac{\text{billion barrels}}{\text{year}}$

$\approx$ 12 billion barrels per year

(10.1) Estimate the average rate of change of oil production from 1900 to 2020. (Hint: This is the slope of the corresponding secant line.)

(10.2) Estimate the instanenous rate of change of oil production at the year 2100. (Hint: This is the derivative at the year 2100.)

(10.3) Compute and interpret $L = \lim_{t\to\infty} Q(t)$.

(10.2) the graph is almost flat @ $t = 2100$, so

$Q'(2100) \approx 0$

(10.3) $\lim_{t\to\infty} Q(t) \approx 2.3$ trillion barrels

Over the entire course of time, from now till eternity, about 2.3 trillion barrels of oil will be produced.