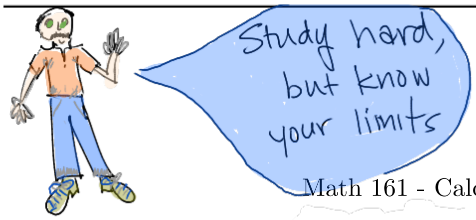

Math 161 - Calculus - Exam 1 - Guide **Name:** Solutions

Show your work to receive full credit.

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1. Evaluate the following limits.:

(1.1) $\lim\limits_{x\to 4} \frac{1}{x} = \frac{1}{4}$

(1.2) $\lim\limits_{x\to 5} 3 = 3$

(1.3) $\lim\limits_{x\to 4} \frac{1}{x-4} =$ DNE b/c $\lim\limits_{x\to 4^-} \frac{1}{x-4} = -\infty$ & $\lim\limits_{x\to 4^+} \frac{1}{x-4} = +\infty$

(1.4) $\lim\limits_{x\to 5} \frac{-1}{(x-5)^2} = -\infty$ $\quad \lim\limits_{x\to 5^+} \frac{-1}{(x-5)^2} = -\infty = \lim\limits_{x\to 5^-} \frac{-1}{(x-5)^2}$

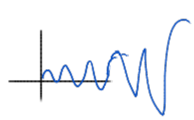
(1.5) $\lim\limits_{x\to 7} \frac{1}{x-4} = \frac{1}{3}$

(1.6) $\lim\limits_{x\to +\infty} \frac{1}{x-4} = 0$

(1.7) $\lim\limits_{x\to +\infty} \frac{\cos(2x)}{x} = 0$ by Squeeze theorem $\quad -\frac{1}{x} \le \frac{\cos 2x}{x} \le \frac{1}{x}$ for all $x \in \mathbb{R}$; $-\frac{1}{x} \to 0$ as $x \to 0$, $\frac{1}{x} \to 0$ as $x \to 0$

(1.8) $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$ b/c $x \approx 0 \Rightarrow \sin(x) \approx x$

(1.9) $\lim\limits_{x\to +\infty} e^x \cos(x)$ DNE



(1.10) $\lim\limits_{x\to 4} \left[\frac{2}{x-4} - \frac{2}{x^2-7x+12}\right] = \lim\limits_{x\to 4} \frac{2(x-3)-2}{(x-4)(x-3)} = \lim \frac{\overset{2(x-4)}{2x-8}}{(x-4)(x-3)} = \lim\limits_{x\to 4} \frac{2}{x-3} = 2$

(1.11) $\lim\limits_{x\to +\infty} x - \sqrt{x^2-6x} \left(\frac{x+\sqrt{x^2-6x}}{x+\sqrt{x^2-6x}}\right) = \lim \frac{x^2-(x^2-6x)}{x+\sqrt{x^2-6x}} = \lim \frac{6x}{x+\sqrt{x^2-6x}}$ [Now ÷ by x]

$\lim \frac{6}{1+\frac{\sqrt{x^2-6x}}{x}} = \lim \frac{6}{1+\sqrt{\frac{x^2-6x}{x^2}}} = \lim\limits_{x\to+\infty} \frac{6}{1+\sqrt{1-\frac{6}{x}}} = \frac{6}{1+1} = 3$

2. (Give a short written response) What does the derivative tell you about a function?

→ instantaneous rate of change

or → how the function is changing @ a single point

or → limit of the average rate of change over an interval as the length of the interval decreases

3. Use the definition of the derivative to compute $f'(x)$.

(3.1) $f(x) = \dfrac{3}{x-1}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{3}{x+h-1} - \frac{3}{x-1}}{h} = \lim \frac{\frac{3(x-1) - 3(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim \frac{\frac{3x - 3 - 3x - 3h + 3}{(x+h-1)(x-1)}}{h} = \lim \frac{\frac{-3h}{(x+h-1)(x-1)}}{h} = \lim_{h\to 0} \frac{-3}{(x+h-1)(x-1)}$$

$$= \frac{-3}{(x+h)^2}$$

(3.2) $f(x) = 5\sqrt{x+2}$

$$\lim_{h\to 0} \frac{5\sqrt{x+h+2} - 5\sqrt{x+2}}{h} = 5 \cdot \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \cdot \left(\frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}\right)$$

$$= 5 \cdot \lim \frac{(x+h+2) - (x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})} = 5 \cdot \lim \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} = \frac{5}{2\sqrt{x+2}}$$

4. Find all solutions

(4.1) $3e^x + 5 = e^x + 11$

$2e^x = 6$ combine like terms

$e^x = 3$

$x = \ln 3$

(4.2) $\left(1 + \frac{0.06}{12}\right)^{2x} = 4$

don't forget to log both sides

$\ln\left(1 + \frac{.06}{12}\right)^{2x} = \ln 4$

$2x \cdot \ln\left(1 + \frac{.06}{12}\right) = \ln 4$

(now divide both sides by what's being multiplied by x)

$x = \dfrac{\ln 4}{2 \cdot \ln\left(1 + \frac{.06}{12}\right)}$

(4.3) $\dfrac{50}{1 + 2e^{3x}} = 10$

step 1. divide both sides by 10

$\dfrac{5}{1 + 2e^{3x}} = 1$

2. cross multiply: $5 = 1 + 2e^{3x}$

3. $4 = 2e^{3x}$

4. $2 = e^{3x}$

5. $\ln 2 = \ln e^{3x} = 3x$

6. $x = \dfrac{\ln 2}{3}$

5. Find $f^{-1}(x)$.

(5.1) $f(x) = \dfrac{1 - 4x}{3x + 2}$

set

1. $y = \dfrac{1 - 4x}{3x + 2}$

swap

2. $x = \dfrac{1 - 4y}{3y + 2}$

solve for y (first cross mult.

3. $(3y + 2)x = 1 - 4y$

4. distribute

$3xy + 2x = 1 - 4y$

5. gather terms containing y

$3xy + 4y = 1 - 2x$

6. factor y

$y(3x + 4) = 1 - 2x$

7. $y = \dfrac{1 - 2x}{3x + 4}$

8. $f^{-1}(x) = \dfrac{1 - 2x}{3x + 4}$

verify: $f \circ f^{-1}(0) = f\left(\frac{1 - 2 \cdot 0}{3 0 + 4}\right) = f\left(\frac{1}{4}\right) = \dfrac{1 - 4(1/4)}{3(1/4) + 2} = \dfrac{1-1}{\sim} = 0.$ ☺ good sign

6. Given

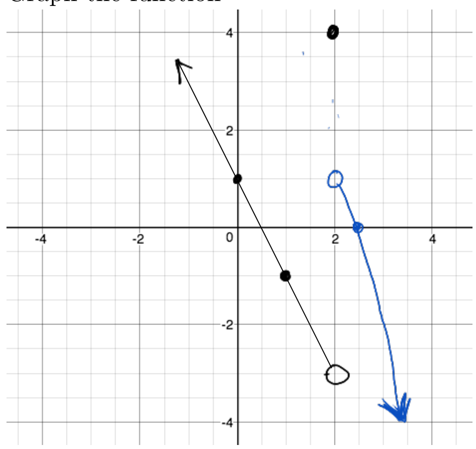
$$f(x) = \begin{cases} 5 - x^2 & x > 2 \\ -2x + 1 & x < 2 \\ 4 & x = 2 \end{cases}$$

For the $5-x^2$ portion:
$x = 2 \Rightarrow 5 - 2^2 = 1$ (open circle @ $(2,1)$)
$y = 0 \Rightarrow 5 - x^2 = 0 \Rightarrow x = \sqrt{5}$

For the $-2x+1$ portion:
$x = 2 \Rightarrow -2(2)+1 = -3$
(open circle @ $(2,-3)$)

For the 4 portion
solid dot @ $(2,4)$

(6.1) Graph the function



(6.2) Finish the definition below:
A function $f(x)$ is continuous at $x = a$ if

$\lim_{x \to a} f(x) = f(a)$
and both exist (meaning they equal some real number)

(6.3) Use the definition of continuity to show that $f(x)$ is not continuous at $x = 2$.

For the $f(x)$ above $\lim_{x \to 2} f(x)$ DNE so $f(x)$ is not cts.

(6.4) Is there a way to define $f(x)$ at $x = 2$ so that $f(x)$ is continuous at $x = 2$? Why or why not?

No For example even if we defined $f(2) = 1$ or $f(2) = -3$, the limit would still not exist.

5. According to Peak Oil Theory, first proposed in 1956, the total amount of crude oil $Q(t)$ produced worldwide up to time t has a graph like the one shown below.

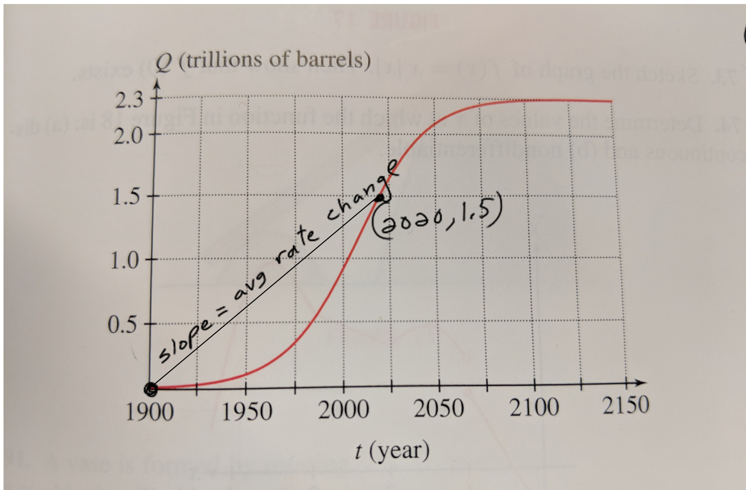


FIGURE 14 Total oil production up to time t.

(6.1) $\dfrac{1.5-0}{2020-1900} = \dfrac{1.5}{120} = \dfrac{3/2}{120}$

$= \dfrac{3}{2} \cdot \dfrac{1}{120} = \dfrac{1}{2} \cdot \dfrac{1}{40} = \dfrac{1}{80}\ \dfrac{tb}{y}$

$= \dfrac{1}{80}\ \dfrac{t}{y} \times \dfrac{1000 \text{ billion}}{1 \text{ trillion}}$

$= \dfrac{1000}{80} = \dfrac{100}{8} = 12.5\ \dfrac{\text{billion barrls}}{\text{yr}}$

(6.2) ≈ 0

(6.3) $\lim_{t \to +\infty} Q(t) = 2.3$

meaning the total amt. of oil ever produced (w/ this model) will be 2.3 trillion barrels

(6.1) Estimate the average rate of change of oil production from 1900 to 2020.

(6.2) Estimate the instanenous rate of change of oil production at the year 2100.

(6.3) Compute and interpret $L = \lim_{t\to\infty} Q(t)$.

7. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height in meters after t seconds is given by $y = 58t - .83t^2$.

   (a) Find the average velocity over the given time intervals:

(7.1) time interval: [1,1.5] 55.925

$$\frac{58(1.5) - .83(1.5)^2 - (58(1) - .83(1)^2)}{1.5 - 1}$$

(7.2) time interval: [1,1.01] 56.3317

$$\frac{58(1.01) - .83(1.01)^2 - (58(1) - .83(1)^2)}{1.01 - 1}$$

(7.3) time interval: [1,1.001] 56.33917

$$\frac{58(1.001) - .83(1.001)^2 - (58(1) - .83(1)^2)}{1.001 - 1}$$

(7.4) Find the instantaneous velocity after one second (to the nearest hundredth).

$y' = 58 - 2(.83)t$

y' when $t = 1$ is $58 - 2(.83)(1) = 56.34$

8. The position of a cat running from a dog down a dark alley is given by the values of the table.

the function →

t (seconds)	0	1	2	3	4	5
s (feet)	0	14	30	73	100	117

Find the average velocity of the cat for the time period beginning with $t = 2$ and lasting

(8.1) 3 seconds time $[2,5] \leadsto \frac{s(5) - s(2)}{5-2} = \frac{117 - 30}{5-2} = \frac{87}{3} = 29$ ft/sec

(8.2) 2 seconds $[2,4] \leadsto \frac{100 - 30}{4-2} = \frac{70}{2} = 35$ ft/sec

(8.3) 1 seconds $[2,3] \leadsto \frac{73 - 30}{3-2} = \frac{43}{1} = 43$ ft/sec

Estimate the instantaneous velocity when $t = 2$ by finding the average velocity from $t = 1$ to $t = 3$.

$$\frac{73 - 14}{3 - 1} = \frac{59}{2} = 29.5 \text{ ft/sec}$$

Do you think this is a good estimate or not? Explain.

poor estimate: from [1,2] the speed is much lower than from [2,3], and is roughly the same as [1,2] (14 ft/s)

It's more likely the inst. velocity is closer to 15 ft/s