Thur wk 6 (3.9)

Implicit Differentiation

1. warm-up
2. derivatives of inverse functions
   a. inverse trig
   b. ln(x)
   c. a^x

warm-up

find y' = dy/dx if $2xy + x + 3y^2 = 17$

$$\frac{d}{dx}\left(2xy + x + 3y^2\right) = \frac{d}{dx}(17)$$

$$\frac{d}{dx}(2xy) + \frac{d}{dx}(x) + \frac{d}{dx}(3y^2) = 0$$

product! power!

$$2\cdot y + 2x\cdot\frac{dy}{dx} + 1 + 6y\cdot\frac{dy}{dx} = 0$$

isolate $\frac{dy}{dx}$

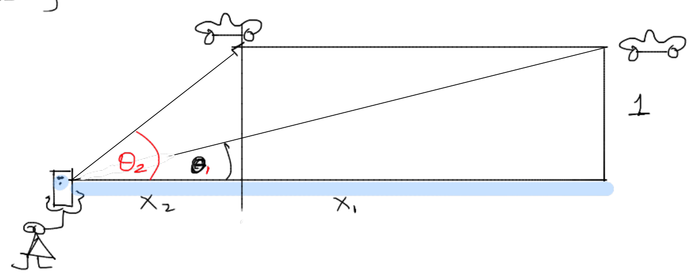
$$\frac{dy}{dx}(2x + 6y) = 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = -1 - 2y$$

$$\boxed{\frac{dy}{dx} = \frac{-1-2y}{2x+6y}}$$

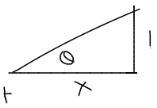
Derivatives of Inverse Functions ($\tan^{-1}(x) = \arctan(x)$)

Motivation

Question: How does angle change as $x \to 0$, (gets closer)



What is θ?



start

$$\tan\theta = \frac{1}{x}$$

isolate θ

$$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{1}{x}\right)$$

$$\theta = \tan^{-1}\left(\frac{1}{x}\right)$$

Answer compute $\displaystyle\lim_{x\to 0} \frac{d}{dx}\left(\tan^{-1}\left(\frac{1}{x}\right)\right)$

(build up to this by

1. get $\frac{d}{dx}\left(\tan^{-1}(u)\right)$
2. sub $u = \frac{1}{x}$

Derivative of arctan(x)

want: $\boxed{\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}}$

① set $y = \tan^{-1}(x)$, goal find y'

② hit w/ tan: $\tan(y) = x$

③ Implicit diff: $\frac{d}{dx}\left(\tan(y)\right) = \frac{d}{dx}(x)$

$\sec^2(y) \cdot y' = 1$

④ solve for y' & get x-back: $\boxed{y' = \frac{1}{\sec^2(y)} = \frac{1}{1+x^2}}$

(*) use: $\tan(y) = x$

$\Longrightarrow$ $\boxed{\tan^2(y) = x^2}$

relationship b/w $\tan^2$ & $\sec^2$

$\frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2}$

$\boxed{\tan^2 y + 1 = \sec^2 y}$

So $x^2 + 1 = \sec^2(y)$

So $\boxed{\frac{d}{dx}\left(\tan^{-1}(u)\right) = \frac{1}{1+u^2} \cdot \frac{du}{dx}}$

to complete motivational question —

$$\lim_{x \to 0} \underbrace{\frac{d}{dx}\left(\tan^{-1}\left(\frac{1}{x}\right)\right)}_{} = \lim_{x \to 0}\left(\frac{-1}{x^2+1}\right) = -1 \ \frac{\text{rad}}{\text{sec}}$$

(dep. speed units

set $u = \frac{1}{x}$, $\frac{du}{dx} = \frac{-1}{x^2}$

$$\parallel$$

$$\frac{1}{1+u^2}\left(\frac{du}{dx}\right) \rightarrow \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{\frac{x^2}{x^2} 1 + \frac{1}{x^2}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{\frac{x^2+1}{x^2}}\left(\frac{-1}{x^2}\right) = \frac{x^2}{x^2+1} \cdot \frac{-1}{x^2} = \frac{-1}{x^2+1}$$

Two other important derivatives

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \dashrightarrow \frac{d}{dx}\left(\ln|u|\right) = \frac{1}{u}\cdot\frac{du}{dx}$$

① set $y = \ln(x)$ ⊛

② find y' by hit w/ $\frac{d}{dx}$ $\qquad \frac{d}{dx}(y) = \frac{d}{dx}\left(\underline{\ln(x)}\right)$ stuck

hit w/ e first!

$$e^{y} = e^{\ln x}$$

$e^y = x$ now hit w/ dx $\qquad \frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(x) = 1$

$$e^y \cdot y' = 1 \Rightarrow y' = \frac{1}{e^y} \overset{⊛}{=} \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Ex $\quad \frac{d}{dx}\left(\ln|x^2|\right) = \frac{1}{x^2}\cdot 2x = \frac{2}{x} = 2\left(\frac{1}{x}\right)$

$\frac{d}{dx}\left(a^x\right)$ when a = constant $\quad \left(\frac{d}{dx}\left(2^x\right)\ ?\right)$

① set $y = a^x$

② hit w/ inverse $\qquad \ln y = \ln a^x \overset{\text{log prop}}{=} x\cdot\ln(a)$

③ hit w/ $\frac{d}{dx}$ $\qquad \frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left(x\cdot \overbrace{\ln(a)}^{\text{just a \#}}\right)$

$$\frac{1}{y}\cdot y' = \ln(a) \Rightarrow y' = y\cdot\ln(a) \Rightarrow y' = a^x \cdot \ln a$$

w/ Chain Rule

$$\boxed{\frac{d}{dx}\left(a^u\right) = a^u \cdot \frac{du}{dx}\,\ln(a)}$$