Exam Review, Chapter 11 Sections 8 - 11

1.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{n=0}^{+\infty} \frac{1}{n!}x^n \text{ for } -\infty < x < +\infty$$

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \text{ for } -\infty < x < +\infty$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}x^{2n} \text{ for } -\infty < x < +\infty$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}x^{2n+1} \text{ for } -1 < x \leq 1$$

2.

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \text{ for } 0 < x \leq 2$$

3.

$$f(x) = \sqrt{x} \simeq 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

$$f^{(5)}(x) = \frac{105}{32}x^{-9/2} \leq \frac{105}{32}(0.64)^{-9/2} = \frac{105 \cdot 5^9}{32 \cdot 4^9} \leq 25 \text{ for } 0.75 \leq x \leq 1.25$$

Note - I used 0 because this was the closest perfect square to the left of 0.80. These calculations should be done without the use of the square root function, after all!

$$\sqrt{1.2} \simeq 1 + \frac{1}{2}(1.2-1) - \frac{1}{8}(1.2-1)^2 + \frac{1}{16}(1.2-1)^3 - \frac{5}{128}(1.2-1)^4 =$$

$$\sqrt{1.2} \simeq 1.0954, \text{ with error less than } \frac{25}{5!}(1.2-1)^5 < 0.000067$$

4. Note that $\pi = 4\tan^{-1}(1)$. So the following will generate the correct digiits of π:

$$4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \cdots$$

Note the 1 is on the far end of the interval of convergence for the Maclaurin series for $\tan^{-1} x$ - so the convergence is about as slow as for the approximation to $\ln 2$ that I showed you in class.

5. Note that if $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n and $|f^{(n+1)}(x)| \leq 3$ for all n and for all $x \in [-1/2, 1/2]$ (using $e < 3$.) Then

$$|R_n(x)| \leq \frac{3}{(n+1)!}\left(\frac{1}{2}\right)^{n+1}$$

When $n = 5$, $|R_n(x)| \leq 0.000066$. (So degree five will work.)

$$\sqrt{e} \simeq 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{1}{24}\left(\frac{1}{2}\right)^4 + \frac{1}{120}\left(\frac{1}{2}\right)^5 \simeq 1.6487$$

6.Note that $1.04 < \pi/3 < 1.06$ and if $f(x) = \sin x$, $f^{(n+1)}(x)$ is either $\sin x$, $-\sin x$, $\cos x$, or $-\cos x$. So $|f^{(n+1)}(x)| \leq 1$ for all n and for all x.

$$|R_n(x)| \leq \frac{1}{(n+1)!}|1.06|^{n+1} \text{ for } -1.06 \leq x \leq 1.06$$

When $n = 9$, $|R_n(x)| \leq 0.0000005$. (So degree nine will work.)

7.

$$\int_0^1 e^{x^2}\,dx \simeq \int_0^1 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\,dx$$

$$\left[x + \frac{1}{3}x^3 + \frac{1}{10}x^5 + \frac{1}{42}x^7\right]\Big|_0^1 = 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} = \frac{51}{35}$$

8.

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \text{ for } -\infty < x < +\infty$$

$$\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} \text{ for } -\infty < x < +\infty$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots = 1$$