some not-so-basic integration-by-parts problems

1.

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \int \frac{2}{3}e^{2x}\sin 3x\,dx$$

$$\begin{array}{lll} u = e^{2x} & \text{and} & dv = \cos 3x\,dx \\ du = 2e^{2x}\,dx & \text{and} & v = \frac{1}{3}\sin 3x \end{array}$$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \left[-\frac{2}{9}e^{2x}\cos 3x - \int -\frac{4}{9}e^{2x}\cos 3x\,dx\right]$$

$$\begin{array}{lll} u = \frac{2}{3}e^{2x} & \text{and} & dv = \sin 3x\,dx \\ du = \frac{4}{3}e^{2x}\,dx & \text{and} & v = -\frac{1}{3}\cos x \end{array}$$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3x\,dx$$

$$\frac{13}{9}\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\int e^{2x}\cos 3x\,dx = \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x + C$$

2.

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$$

$$u = \sec\theta \qquad dv = \sec^2\theta\,d\theta$$

$$du = \sec\theta\tan\theta \qquad v = \tan\theta$$

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\left(\sec^2\theta - 1\right)\,d\theta$$

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

$$2\int \sec^3\theta\,d\theta = \sec\theta\tan\theta + \int \sec\theta\left(\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\right)\,d\theta$$

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\tan\theta+\sec\theta}\,d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\tan\theta+\sec\theta| + C$$

3.

$$\int 3x^2 \sin^{-1} x \, dx = x^3 \sin^{-1} x - \int \frac{x^3}{\sqrt{1-x^2}} \, dx$$

$$\begin{array}{lcl} u = \sin^{-1} x & \text{and} & dv = 3x^2 \, dx \\ du = \frac{1}{\sqrt{1-x^2}} \, dx & \text{and} & v = x^3 \end{array}$$

Note that

$$\int \frac{x^3}{\sqrt{1-x^2}} \, dx = \int \frac{1}{2} \cdot \frac{1-u}{\sqrt{u}} \, du = \frac{1}{2} \int u^{-1/2} - u^{1/2} \, du$$

$$= u^{1/2} - \frac{1}{3} u^{3/2} + C$$

$$= \left(1-x^2\right)^{1/2} - \frac{1}{3}\left(1-x^2\right)^{3/2} + C$$

So

$$\int 3x^2 \sin^{-1} x \, dx = x^3 \sin^{-1} x - \int \frac{x^3}{\sqrt{1-x^2}} \, dx$$

$$= x^3 \sin^{-1} x - \left(1-x^2\right)^{1/2} + \frac{1}{3}\left(1-x^2\right)^{3/2} + C$$