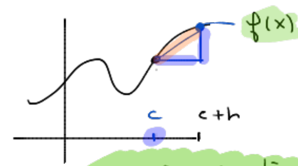
Thursday

1. Calculus I Review
    a. What is Calculus I?
        i. Precise way of calculating change (in functions)
        ii. Two Branches: Differential & Integral
            1. Both depend on the idea of an infinite limit

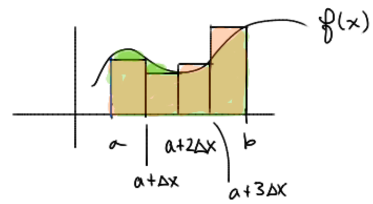


The Derivative

$$\boxed{f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}}$$

Link: https://www.desmos.com/calculator/t6sxoyxyty

The Definite Integral



w/ $\Delta x = \frac{b-a}{n}$

$$\boxed{\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*) \cdot \Delta x}$$

Link: https://www.desmos.com/calculator/tgyr42ezjq

Note: Differentiating is "easier" than integrating

eg:

$$\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \cdot 2x$$

$$\int e^{x^2}\,dx = \text{NO elementary sol'n}$$

unless.. our function is given discrete data

Basic Anti-Derivative Table

Functions	Anti-Derivatives
x^n	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$-\csc(x)\cot(x)$	$\csc(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}(x)$

Q: why is $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$

① set $\boxed{y = \sec^{-1} x}$, goal $\frac{dy}{dx} = y'$.

② So $\sec(y) = \sec(\sec^{-1}(x)) = x$

outside

inside

$$\boxed{\sec(y) = x}$$

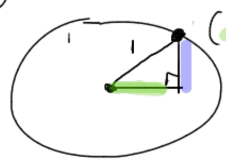
③ Implicit Differentiation $\longrightarrow$ $\underbrace{\frac{d}{dx}(\sec(y))}_{\sec(y)\tan(y)\cdot y'} = \frac{d}{dx}(x)$

$$\boxed{\text{chain rule: } \frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)}$$

$\sec(y)\tan(y)\cdot y' = 1$

④ $y' = \frac{1}{\sec(y)\tan(y)} = \frac{1}{x\cdot\tan(y)} = \boxed{\frac{1}{x\sqrt{x^2-1}}}$

⑤ rewrite $\tan(y)$ in terms of $\sec(y)$



$(\cos(y), \sin(y))$

Pyth Thm: $\frac{\cos^2(y)}{\cos^2(y)} + \frac{\sin^2(y)}{\cos^2(y)} = \frac{1}{\cos^2 y}$

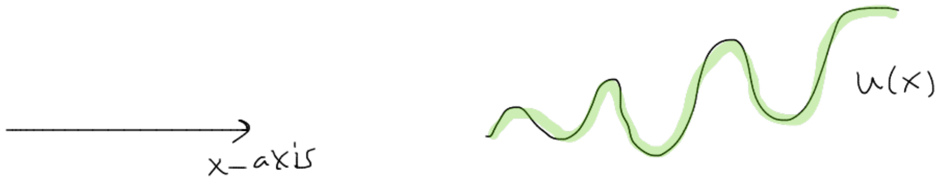
$$1 + \tan^2(y) = \sec^2 y$$

$$\tan(y) = \sqrt{\sec^2 y - 1}$$

So $\tan(y) = \sqrt{x^2-1}$

u-sub

In this fact: $\int x^n \, dx = \frac{x^{n+1}}{n+1}$ the only requirement is that x be continuously varying real variable



True that $\int u^n \, du = \frac{u^{n+1}}{n+1}$ where $u = f(x)$

Ex. $\int (3x^2+1)^5 \cdot x \, dx$

Note: there's no product rule for integration.

Idea: Partition integrand into two pieces where one is (a multiple of) the derivative of the other.

$\boxed{u = 3x^2+1}$

$\frac{d}{dx}(u) = \frac{d}{dx}(3x^2+1)$

$\frac{du}{dx} = 6x$ so $\boxed{du = 6x \, dx}$ ⟶ $\boxed{\frac{1}{6x} \cdot du = dx}$

treat like fraction

$= \int u^5 \cancel{x} \frac{1}{6\cancel{x}} du = \frac{1}{6} \int u^5 \, du$

$= \frac{1}{6} \cdot \frac{u^6}{6} + C$

$= \boxed{\frac{(3x^2+1)^6}{36} + C}$

Why is it that

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad ?$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

For $f(x) = x^n$

expand!

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim \frac{(x+h)^n - x^n}{h}$$

$$= \lim \frac{\boxed{\overset{=1}{\binom{n}{0}}x^n} + \binom{n}{1}x^{n-1}\cdot\cancel{h} + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n \boxed{-x^n}}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{nx^{n-1} + \overbrace{\binom{n}{2}x^{n-2}h + \dots}^{\text{stuff w/ h term}} \to 0}{1} = nx^{n-1}$$

Derivatives:

Power Rule:

$f(x) = x^n$, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^n - x^n}{h}$

$$= \lim_{h\to 0} \frac{\binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n}x^0}{h} = \frac{\binom{n}{1}x^{n-1}h}{h} = nx^{n-1}$$

The pink terms are equal

The white terms vanish as $h \to 0$.

TRIG.

$f(x) = \sin(x)$ $\quad f'(x) = \lim_{h\to 0} \dfrac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \dfrac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$

$$= \lim_{h\to 0} \sin(x) \cdot \left(\frac{\cos(h) - 1}{h}\right) + \cos(x) \lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)$$

$$= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)$$

These ideas lead to these formulas:

Functions	Derivatives
x^n	nx^{n-1}
$\ln(x)$	$1/x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$
e^x	e^x
$\sin^{-1}(x)$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\dfrac{1}{1+x^2}$
$\sec^{-1}(x)$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\csc^{-1}(x)$	$\dfrac{-1}{x\sqrt{x^2-1}}$
$\cot^{-1}(x)$	$\dfrac{-1}{1+x^2}$

Functions	Anti-Derivatives
x^n	$\dfrac{x^{n+1}}{n+1}$
$1/x$	$\ln\lvert x\rvert$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\csc(x)\cot(x)$	$-\csc(x)$
$\csc^2(x)$	$-\cot(x)$
e^x	e^x
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\dfrac{1}{\sqrt{1-x^2}}$	$-\cos^{-1}(x)$
$\dfrac{1}{1+x^2}$	$\tan^{-1}(x)$
$\dfrac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}(x)$
$\dfrac{1}{x\sqrt{x^2-1}}$	$-\csc^{-1}(x)$
$\dfrac{-1}{1+x^2}$	$\cot^{-1}(x)$

why is $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$?

① set $y = \sin^{-1}(x)$. Goal, find $y' = \frac{dy}{dx}$.

(chain rule)

② so $\sin(y) = x$ & $\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$ so $\cos(y) \cdot \frac{dy}{dx} = 1$

③ $\frac{dy}{dx} = \frac{1}{\cos(y)}$ & since $\sin^2(y) + \cos^2(y) = 1$ it follows that $\cos(y) = \sqrt{1 - \sin^2(y)}$
w/ $\sin(y) = x$ from step ②

④ $y' = \frac{1}{\sqrt{1-x^2}}$

u- substitution

In this fact $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ there's nothing special about x. (what really matters is that it's a continuously changing real variable)

So if $u = f(x)$ then $\frac{du}{dx} = f'(x)$ or $du = f'(x)\,dx$. Meaning

$\int (x^2+7)^5 \cdot (2x)\,dx$ can be solved by

$u = x^2+7$, $\frac{du}{dx} = 2x$ or $du = 2x\,dx$ so the integral is transformed into

$$\int u^5\,du = \frac{u^6}{6} + C$$

Another view on

$$\int \cos(x)\,dx = \sin(x)$$

LINK: https://www.desmos.com/calculator/aszy5en1qt