Taylor Remainders . . . .

Taylor Inequality: Given a Taylor polynomial of degree n about a, if $|f^{(n+1)}(x)| \leq M$ for all x such that $|x-a| < d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x such that $|x-a| < d$.

1. The Maclaurin series for $\cos x$ is below. Use it to derive the Maclaurin series for $\sin x$.

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

2. Using Taylor's inequality, determine what degree polynomial you need to use to estimate $\sin(0.3)$ to within 0.001 of the actual value. Calculate the estimate.

Use $a = 0$ and $M = 1$. Since $|f^{(k)}(x)| \leq 1$ for all real numbers x, we don't need to worry about selecting d. We need n such that

$$\frac{1}{(n+1)!}|.3|^{n+1} \leq 0.001$$

Note that $n = 3$ is the lowest value that guarantees this level of accuracy.

$$\sin(.3) \simeq 0.3 - \frac{1}{3!}.3^3 = 0.2955$$

3. Use the fifth degree Taylor polynomial for $\sin x$ to estimate $\sin 1$. Use Taylor's Inequality to to obtain upper and lower bounds fro the real value of $\sin 1$ based on your estimate.

$$\sin 1 \simeq 1 - \frac{1}{3!} + \frac{1}{5!} \simeq 0.841667$$

$$|R_5(1)| \leq \frac{1}{6!} \simeq 0.001389$$

So

$$0.841667 - 0.001389 = 0.840278 < \sin 1 < 0.843056 = 0.841667 + 0.001389$$