Taylor Remainders . . . .

1. Use the Maclaurin series for e^x and the Taylor Remainder Theorem to estimate $e^{0.1}$ to within .0001 of the acutal value.

Taylor's Inequality:

$$\text{If } \left|f^{(n+1)}(x)\right| \leq M \text{ for all } x \text{ such that } |x-a| < d,$$

$$\text{then } |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a| < d.$$

First consider the function $f(x) = e^x$. We'll use the Maclaurin series for e^x, so $a = 0$. (The Maclaurin series is easy to produce, and 0.1 is close to 0 - these are both reasons to use the Maclaurin series for e^x.)

Now think about all of the derivatives of $f(x) = e^x$. This is a special case - all of the derivativss are e^x. This makes e^x a bit easier to work with than other functions. One less out of all of the "moving targets" when trying to figure out how long a polynomial we need to get an estimate within a given error bound.

Since we cannot establish an upper bound for $|f^{(n)}(x)|$ for all reals, we need to restrict our "window." That is, we need to select d and an associated M so that for any x such that $|x| \leq d$, then $|f^{(n)}(x)| \leq M$. Since we are interested in estimating $e^{0.1}$, we also need $|0.1| \leq d$.

If we let $d = 1$, then an upper bound for $|e^x|$ on $[-1, 1]$ is e, which is less than 3. Note that we could let $d = 0.1$, but then the upper bound on $|e^x|$ on $[-0.1, 0.1]$ is $e^{0.1}$, and this is what we are trying to estimate. You need to act as if the function e^x has been removed from your calculator - we are building it from scratch. We'll assume that e is less than 3.

From the inequality above, we know

$$|R_n(0.1)| \leq \frac{3}{(n+1)!}|0.1|^{n+1}$$

If we set

$$\frac{3}{(n+1)!}|0.1|^{n+1} \leq 0.0001$$

and solve for n, this will tell us what degree polynomial we need (or simply where we can stop in the series).

The smallest positive integer n that satisfies the inequality above is $n = 3$.

The Maclaurin series for e^x is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{+\infty} \frac{1}{n!}x^n$$

So

$$e^{0.1} \simeq 1 + 0.1 + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 = \frac{6631}{6000} \simeq 1.10517$$

2. Use the Maclaurin series for e^x and the Taylor Remainder Theorem to estimate e^2 to within .0001 of the acutal value.

Same issues as before, but now when we select d it must be at least as big as 2. If $d = 2$, then the upper bound of e^x (and all of its derivatives) on $[-2, 2]$ is e^2. We are allowed to use $e < 3$, so we can let $M = 9$.

From the inequality above, we know

$$|R_n(2)| \leq \frac{9}{(n+1)!}|2|^{n+1}$$

If we set

$$\frac{9}{(n+1)!}|2|^{n+1} \leq 0.0001$$

and solve for n, this will tell us what degree polynomial we need (or simply where we can stop in the series).

The smallest positive integer n that satisfies the inequality above is $n = 11$.

$$e^2 \simeq 1 + 2 + \frac{1}{2!}2^2 + \frac{1}{3!}2^3 + \cdots + \frac{1}{11!}2^{11} \simeq 7.38905$$