Estimation Practice

1. Find the value of n which will give an estimate of the definite integral below to within 0.001 of the actual value using the indicated method.

$$\int_0^5 \sin(x^2)\,dx$$

You can use:

$f(x) = \sin(x^2)$

$f'(x) = 2x\cos(x^2)$

$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$

$$\begin{aligned} f^{(3)}(x) &= -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2) \\ &= -12x\sin(x^2) - 8x^3\cos(x^2) \end{aligned}$$

$$\begin{aligned} f^{(4)}(x) &= -12\sin(x^2) - 24x^2\cos(x^2) - 24x^2\cos(x^2) + 16x^4\sin(x^2) \\ &= -12\sin(x^2) - 48x^2\cos(x^2) + 16x^4\sin(x^2) \end{aligned}$$

a. Trapezoidal

Note that for all x in $[0,5]$, $|f''(x)| \leq 2 + 4(5^2) = 102$. We want

$$|E_T| \leq \frac{102(5-0)^3}{12n^2} \leq 0.001$$

$$\frac{102(5-0)^3}{12(0.001)} \leq n^2 \implies n = 1031$$

b. Midpoint

We can use the same K value of 102 from part a. We want

$$|E_M| \leq \frac{102(5-0)^3}{24n^2} \leq 0.001$$

$$\frac{102(5-0)^3}{24(0.001)} \leq n^2 \implies n = 729$$

c. Simpson's Rule

Note that for all x in $[0,5]$, $\left|f^{(4)}(x)\right| \leq 12 + 48(5^2) + 16(5^4) = 11212$. We want

$$|E_S| \leq \frac{11212(5-0)^5}{180n^4} \leq 0.001$$

$$\frac{11212(5-0)^5}{180(0.001)} \leq n^4 \Longrightarrow n = 120$$

(Remember that for Simpson's Rule, n has to be even . . . .)

2. Use integrals to estimate the series below to within 0.001 of the actual value.

$$\sum_{n=1}^{+\infty} \frac{1}{n^5}$$

From the integral test, we have that

$$s_n + \int_{n+1}^{+\infty} \frac{1}{x^5}\,dx \le \sum_{n=1}^{+\infty} \frac{1}{n^5} \le s_n + \int_{n}^{+\infty} \frac{1}{x^5}\,dx$$

The difference in the upper and lower bound is

$$\int_{n}^{+\infty} \frac{1}{x^5}\,dx - \int_{n+1}^{+\infty} \frac{1}{x^5}\,dx = \int_{n}^{n+1} \frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_n^{n+1}$$

$$= -\frac{1}{4\,(n+1)^4} - \left(-\frac{1}{4n^4}\right) = \frac{(n+1)^4 - n^4}{4(n+1)^4 n^4}$$

Note that

$$\frac{(n+1)^4 - n^4}{4(n+1)^4 n^4} < 0.001 \text{ for } n = 4$$

a few calculations:

$$s_4 = 1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024}$$

$$\int_{4}^{+\infty} \frac{1}{x^5}\,dx = \lim_{b\to+\infty}\left[\int_4^b \frac{1}{x^5}\,dx\right] = \lim_{b\to+\infty}\left[-\frac{1}{4x^4}\right]_4^b$$

$$= \lim_{b\to+\infty}\left[-\frac{1}{4b^4} - \left(-\frac{1}{4(4^4)}\right)\right] = \frac{1}{1024}$$

$$\int_{5}^{+\infty} \frac{1}{x^5}\,dx = \lim_{b\to+\infty}\left[\int_5^b \frac{1}{x^5}\,dx\right] = \lim_{b\to+\infty}\left[-\frac{1}{4x^4}\right]_5^b$$

$$= \lim_{b\to+\infty}\left[-\frac{1}{4b^4} - \left(-\frac{1}{4(5^4)}\right)\right] = \frac{1}{2500}$$

So

$$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{2500} \le \sum_{n=1}^{+\infty}\frac{1}{n^5} \le 1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{1024}$$

3. Show that the following series converges by the Alternating Series Test. Then estimate the value of the series within 0.001 of the actual value.

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^5}$$

Converges by Alternating Series Test:

1. $\lim_{n \to +\infty} \frac{1}{n^5} = 0$

2. $\frac{d}{dx}\left[\frac{1}{x^5}\right] = \frac{-5}{x^6} < 0$ for all $x > 0$,

$$\text{so } \left\{\frac{1}{n^5}\right\}_{n=1}^{+\infty} \text{ is a strictly decreasing sequence.}$$

Note that $1/4^5 < 0.001$ - so we can stop adding up terms at $n = 4$.

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^5} \simeq -1 + \frac{1}{32} - \frac{1}{243} + \frac{1}{1024} = -\frac{241837}{248832}$$

4. Estimate the given indefintie integral using the indicated method . . .

$$\int_0^4 x^2\,dx$$

(a) Using Trapezoidal Rule with $n = 2$.

$$S_{T,2} = 2\left[\frac{f(0)+f(2)}{2}\right] + 2\left[\frac{f(2)+f(4)}{2}\right] =$$

$$S_{T,2} = 2\left[\frac{0+4}{2}\right] + 2\left[\frac{4+16}{2}\right] = 24$$

(b) Using Midpoint Rule with $n = 2$.

$$S_{M,2} = 2f(1) + 2f(3) =$$

$$S_{M,2} = 2\cdot 1 + 2\cdot 9 = 20$$

(c) Using Simpson's Rule with $n = 4$.

$$S_{S,4} = \frac{1}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right] =$$

$$S_{S,4} = \frac{1}{3}\left[0 + 4\cdot 1 + 2\cdot 4 + 4\cdot 9 + 16\right] = \frac{64}{3}$$

also calculate it this way:

$$S_{S,4} = \frac{1}{3}S_{T,2} + \frac{2}{3}S_{M,2} =$$

$$S_{S,4} = \frac{1}{3}(24) + \frac{2}{3}(20) = \frac{64}{3}$$

(d) Now just calculate the value directly:

$$\int_0^4 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^4 = \frac{64}{3}$$