Estimation Practice

2. Use integrals to estimate the series below to within 0.001 of the actual value.

$$\sum_{n=1}^{+\infty} \frac{1}{n^5}$$

From the integral test, we have that

$$s_n + \int_{n+1}^{+\infty} \frac{1}{x^5}\,dx \le \sum_{n=1}^{+\infty} \frac{1}{n^5} \le s_{n+1} + \int_{n+1}^{+\infty} \frac{1}{x^5}\,dx$$

The difference in the upper and lower bound is a_{n+1}.
Note that

$$\frac{1}{(n+1)^5} < 0.001 \text{ for } n = 3$$

a few calculations:

$$s_3 = 1 + \frac{1}{32} + \frac{1}{243}$$

$$s_4 = 1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024}$$

$$\int_4^{+\infty} \frac{1}{x^5}\,dx = \lim_{b\to+\infty}\left[\int_4^b \frac{1}{x^5}\,dx\right] = \lim_{b\to+\infty}\left[-\frac{1}{4x^4}\right]_4^b$$

$$= \lim_{b\to+\infty}\left[-\frac{1}{4b^4} - \left(-\frac{1}{4(4^4)}\right)\right] = \frac{1}{1024}$$

So

$$1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024} \le \sum_{n=1}^{+\infty} \frac{1}{n^5} \le 1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024} + \frac{1}{1024}$$