MA 163 Exam 3

1. Find the third degree Taylor polynomial for $x^{3/2}$ about $x = 1$.

$$f(x) = x^{3/2} \qquad f(1) = 1$$

$$f'(x) = \tfrac{3}{2}x^{1/2} \qquad f'(1) = \tfrac{3}{2}$$

$$f''(x) = \tfrac{3}{4}x^{-1/2} \qquad f''(1) = \tfrac{3}{4}$$

$$f^{(3)}(x) = -\tfrac{3}{8}x^{-3/2} \qquad f^{(3)}(1) = -\tfrac{3}{8}$$

Using the above calculations (and don't forget the factorials) . . . .

$$x^{3/2} \approx 1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{1}{16}(x-1)^3$$

2. Find the arc length of the curve

$$y = \frac{2}{3}x^{\frac{3}{2}} \text{ from 0 to 3}$$

Since the derivative is just $\sqrt{x}$

$$l = \int_0^3 \sqrt{1+(f'(x))^2}\,\mathrm{d}x = \int_0^3 \sqrt{(1+x)}\,\mathrm{d}x = \int_1^4 \sqrt{(u)}\,\mathrm{d}x$$

$$= \frac{2u^{3/2}}{3}\Big|_1^4 = \frac{2}{3}(4^{3/2} - 1^{3/2}) = \frac{2}{3}(8-1) = \tfrac{14}{3}$$

3. Evaluate the integral:

$$\int_7^{+\infty} \frac{1}{4x^3}\,dx = \lim_{b\to+\infty}\int_7^b \frac{1}{4}x^{-3}\,dx = \lim_{b\to+\infty}\left[-\frac{1}{8}x^{-2}\right]_7^b$$

$$= \lim_{b\to+\infty}\left[-\frac{1}{8b^2} - \left(-\frac{1}{392}\right)\right] = \frac{1}{392}$$

So . . .

$$\frac{1}{4(4^3)}+\frac{1}{4(5^3)}+\frac{1}{4(6^3)}+\frac{1}{392} \le \sum_{n=4}^{+\infty}\frac{1}{4n^3} \le \frac{1}{4(4^3)}+\frac{1}{4(5^3)}+\frac{1}{4(6^3)}+\frac{1}{4(7^3)}+\frac{1}{392}$$

$$0.00961 \le \sum_{n=4}^{+\infty}\frac{1}{4n^3} \le 0.0103$$

4. The Maclaurin series for $\cos x$ is below.

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

Find the interval of convergence.

$$\frac{\left|a_{n+1}\right|}{\left|a_n\right|}=\frac{\frac{|x|^{2(n+1)}}{(2(n+1))!}}{\frac{|x|^{2n}}{(2n)!}}=\frac{(2n)!}{(2n+2)!}\cdot\frac{|x|^{2n+2}}{|x|^{2n}}$$

$$=\frac{(2n)!}{(2n+2)(2n+1)(2n)!}|x|^2=\frac{1}{(2n+2)(2n+1)}|x|^2$$

$$\lim_{n\to+\infty}\frac{\left|a_{n+1}\right|}{\left|a_n\right|}=\lim_{n\to+\infty}\frac{1}{(2n+2)(2n+1)}|x|^2$$

$$=|x|^2\lim_{n\to+\infty}\frac{1}{(2n+2)(2n+1)}=|x|^2\cdot 0=0<1$$

By the Ratio Test, this series converges for all values of x.

5. Use the series above to obtain an estimate for $\cos\left(\frac{1}{2}\right)$ to within 0.01 of the actual vallue.

$$\cos\left(\frac{1}{2}\right)=1-\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{24}\left(\frac{1}{2}\right)^4\simeq 0.8776$$

(Alternating series, and the last fraction is less than 0.01, so we can "stop" here.)

6. Use an eighth degree Taylor polynomial to estimate

$$\int_0^1\frac{\cos(x^2)-1}{x}\,dx\simeq\int_0^1-\frac{1}{2}x^3+\frac{1}{24}x^7\,dx=\left[-\frac{1}{8}x^4+\frac{1}{192}x^8\right]_0^1=-\frac{1}{8}+\frac{1}{192}=-\frac{23}{192}$$

$$\left(\cos(x^2)\simeq 1-\frac{1}{2}\left(x^2\right)^2+\frac{1}{24}(x^2)^4=1-\frac{1}{2}x^4+\frac{1}{24}x^8\Longrightarrow\frac{\cos(x^2)-1}{x}\simeq-\frac{1}{2}x^3+\frac{1}{24}x^7\right)$$

7. The Maclaruin series for the funciton $\ln(x+1)$ is below. Find the interval of convergence.

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{(-1)^{n+1+1}}{n+1} x^{n+1}\right|}{\left|\frac{(-1)^n}{n+1} x^n\right|} = \frac{n}{n+1} \cdot \frac{|x|^{n+1}}{|x|^n} = \frac{n}{n+1}|x|$$

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{n}{n+1}|x| = |x| \lim_{n\to+\infty} \frac{n}{n+1} = |x| \cdot 1 = |x|$$

From the Ratio Test, we know that the series converges for $-1 < x < 1$ and diverges for $(-\infty, -1) \cup (1, +\infty)$. We have to check whether or not the series converges for $x = 1$ and $x = -1$.
When $x = 1$, the power series is equal to

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ which converges}$$

When $x = -1$, the power series is equal to

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots \text{ which diverges}$$

Therefore, the interval of convergence for the Maclaurin Series for $\ln(x+1)$ is $(-1, 1]$.

8. Use # 8 to find a series that converges to $\ln 2$.

$$\text{Plug } x = 1 \text{ into the series in \#8: } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

9. Prove $e^{i\pi}+1=0$ We know

$$e^x=\sum_{n=0}^{+\infty}\frac{x^n}{n!}$$

$$\sin(x)=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x)=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

so

$$e^{i\pi}=\sum_{n=0}^{+\infty}\frac{(i\pi)^n}{n!}=1+\frac{i\pi}{1}+\frac{(i\pi)^2}{2!}+\frac{(i\pi)^3}{3!}+\frac{(i\pi)^4}{4!}+\frac{(i\pi)^5}{5!}+\frac{(i\pi)^6}{6!}+\dots$$

since the series converges we can rearrange the order of the terms, keeping the even powers separate from the odd

$$e^{i\pi}=\sum_{n=0}^{+\infty}\frac{(i\pi)^n}{n!}=1+\frac{(i\pi)^2}{2!}+\frac{(i\pi)^4}{4!}+\frac{(i\pi)^6}{6!}+\dots+\frac{i\pi}{1}+\frac{(i\pi)^3}{3!}+\frac{(i\pi)^5}{5!}+\dots$$

using the fact that $i^2=-1$ and $i^4=1$ we get

$$e^{i\pi}=\sum_{n=0}^{+\infty}\frac{(i\pi)^n}{n!}=1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\dots+\frac{i\pi}{1}-\frac{i(\pi)^3}{3!}+\frac{i(\pi)^5}{5!}+\dots$$

factoring out i in the second half

$$e^{i\pi}=\sum_{n=0}^{+\infty}\frac{(i\pi)^n}{n!}=1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\dots+i\left(\frac{\pi}{1}-\frac{\pi^3}{3!}+\frac{\pi^5}{5!}+\dots\right)$$

reconizing the odd powers are Maclaurin series for cosine evaluated at $x=i\pi$ and likewise for the odds and sine.

$$e^{i\pi}=\cos(\pi)+i\sin(\pi)=-1+0=-1$$

thus

$$e^{i\pi}+1=0$$