Friday - Week 10

From last time (see #10 WeBWorK)

Give a power series for $f(x) = x^{.8}$ centered at 5

Taylor Series @ $a=5$

$$f(x) = \sum c_n (x-5)^n$$

$f(x) = x^{.8}$

$f'(x) = .8x^{.8-1}$

$f''(x) = (.8)(.8-1)x^{.8-2}$

$f'''(x) = (.8)(.8-1)(.8-2)x^{.8-3}$

$\vdots$

$f^{(n)}(x) = (.8)(.8-1)(.8-2)\cdots(.8-(n-1))x^{.8-n}$

$f(5) = 5^{.8}$

$f'(5) = .8(5)^{.8-1}$

$f''(5) = .8(.8-1)(5)^{.8-2}$

$f'''(5) = (.8)(.8-1)(.8-2)\cdot 5^{.8-3}$

$\vdots$

$$\boxed{f^{(n)}(5) = .8(.8-1)(.8-2)\cdots(.8-(n-1))5^{.8-n}}$$

Int. of Conv

Start

$$\sum \frac{f^{(n)}(5)(x-5)^n}{n!} = \sum \frac{.8(.8-1)\cdots(.8-\overset{(n+1)}{(n-1)})5^{.8-n}(x-5)^n}{n!}$$

$.8-(n+1-1)$

Ratio Test

Notation $\prod_{k=0}^{n+1}(.8-k)$

$5^{.8-n-1} = \boxed{5^{.8-n}\cdot 5^{-1}}$

$$\left| \frac{.8(.8-1)\cdots(.8-((n+1)-1))\cdot \boxed{5^{.8-(n+1)}}\cdot(x-5)^{n+1}}{(n+1)!} \cdot \frac{n!}{\prod_{k=0}^{n}(.8-k)\, 5^{.8-n}\cdot(x-5)^n} \right|$$

$$\left| \frac{(.8-n)\cancel{5^{.8-n}}\cdot 5^{-1}(x-5)}{(n+1)\cancel{5^{.8-n}}} \right| = \left| \frac{(n-.8)5^{-1}(x-5)}{n+1} \right| \xrightarrow[n\to\infty]{} |5^{-1}\cdot(x-5)|$$

$$= \left|\frac{x-5}{5}\right| < 1.$$

$|x-5| < 5$

$-5 < x-5 < 5$

$0 < x < 10$

so $x \in (0,10)$

Theory of binomial series $r = .8 > 0$ include endpoints in interval of convergence → $\boxed{[0,10]}$

≥ 0 both converge

$-1 \leq r < 0$ 1 converge -1 diverge

$-1 > r$ both diverge

Taylor's Remainder Theorem

"How many terms in the approximating poly are needed to get a close estimate".

Start: Taylor series for $f(x)$ centered at a!

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}}_{\substack{\text{degree 2 Taylor Poly} \\ P_2(x)}} + \underbrace{\frac{f'''(a)(x-a)^3}{3!} + \cdots}_{\substack{\text{remainder} \\ R_2(x)}}$$

$R_2(x) = f(x) - P_2(x)$

= error by approximating w/ $P_2(x)$

goal: make $R_2(x)$ small.

Taylor: If $|f^{(n+1)}(x)| \leq M$ (some constant) for all x near a s.t. $|x-a| < d$ (window size)

then $|R_n(x)| \leq \dfrac{M \cdot |x-a|^{n+1}}{(n+1)!}$ for all x s.t. $|x-a| < d$

Find some n for which ↑ is small

Ex Estimate $e^{.1}$ to within $.0001$ of the actual value.

How?

Think: e^x, find a Taylor Series centered close to $.1$. — 0 is good idea.

MacLaurin Series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

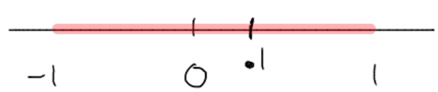
Look at Remainder Thm:

If $|f^{(n+1)}(x)| \le M$ for all x in window about a.

then $|R_n(x)| \le \dfrac{M\,|x-a|^{n+1}}{(n+1)!}$

but for us: $a=0$

$$|R_n(x)| \le \frac{M|x|^{n+1}}{(n+1)!}$$



$f^{(n)}(x) = e^x$

window $(-1,1)$

what numbers $\geq e^x$ when $x \in (-1,1)$?

Max @ $x=1 \Rightarrow e < 3$

set $M = 3$

Recap $|f^{(n+1)}(x)| = |e^x| < 3$ for $x \in (-1,1)$

we apply Taylor's Thm!

$$|R_n(x)| \le \frac{3|x|^{n+1}}{(n+1)!}$$

since we're approximating $e^{.1}$ — plug in $.1$

$$R_n(.1) \le \frac{3|.1|^{n+1}}{(n+1)!}$$

trial/error
$n=1$
$n=2$
$n=3$ ⟶ gives $|R_n(.1)| < .00001 < .0001$ ☺

(1 point) **Library/ma123DB/set13/s11_12_16.pg**

Find $T_5(x)$, the degree 5 Taylor polynomial of the function $f(x) = \cos(x)$ at $a = 0$.

$T_5(x) =$

Find all values of x for which this approximation is within 0.004942 of the right answer. Assume for simplicity that we limit ourselves to $|x| \le 1$.

$|x| \le$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$T_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$R_5(x)$, requires $f^6(x) \le 1$

b/c cos & sin are always ≤ 1

$$|R_5(x)| \le \frac{1 \cdot |x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!} < .004942$$

Bigger than Error

$$|x|^6 < .004942 \cdot 6!$$

solve for

$$|x| < 1.2$$

$$|x| \le 1$$