Ex Approximate $\ln(1.3)$ up to 10^{-3}.

since $1.3 \approx 1$, use Taylor series @ $a=1$. (could we use $a=0$? yes, just need more "n" terms.)

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$f'' = \frac{-1}{x^2}$

$f''' = \frac{2}{x^3}$

$f'''' = \frac{-6}{x^4}$

$\vdots$

$f^{(n)}(x) = (-1)^n \frac{(n-1)!}{x^n}$

$f^{(n+1)}(x) = \frac{(-1)^{n+1} \cdot n!}{x^{n+1}}$

@ $a=1$

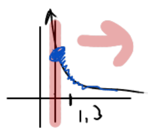
$f'(1) = 1$ $n=0$

$f'' = -1$

$f''' = 2$

$f'''' = -6$

$\vdots$



$$\sum_{n=1}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}$$

$$= (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!}$$

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

$|R_n(x)| \le M \frac{|x-1|^{n+1}}{(n+1)!}$, $x = 1.3$, $M \geq f^{n+1}$

the only way to bound this particular derivative is to restrict the interval away from 0. remember to include your 1.3.

restrict to $[1, \infty)$: $M = \frac{n!}{1^{n+1}} = n!$

b/c $f^{n+1}(x)$ is max @ $x=1$

$$\frac{n! \, \overbrace{|1.3-1|}^{0.3}{}^{n+1}}{\underbrace{(n+1)!}_{=(n+1)n!}} < 10^{-3}$$

solve

$$\Rightarrow 10^3 < \frac{n+1}{(0.3)^{n+1}}$$

$$(0.3)^{n+1} = \left(\frac{3}{10}\right)^{n+1}$$

$$10^3 < (n+1)\left(\frac{10}{3}\right)^{n+1} < (n+1)\Big(\underbrace{\frac{12}{3}}_{=4}\Big)^{n+1}$$

$n=4$

$10^3 < (n+1)4^{n+1}$

$\ldots < 5 \cdot 4^5$ ✓

sub $x = 1.3$

$$(x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{2} - \frac{(x-1)^4}{3!}$$

$$\ln(1.3) \approx 0.3 - \frac{(0.3)^2}{2} + \frac{(0.3)^3}{3} - \frac{(0.4)^4}{4}$$

$$|R_n(x)|$$
$$\|$$
error = remainder

thm

$|R_n(x)| \leq \dfrac{M \cdot |x-a|^{n+1}}{n+1}$ for $x \in (b,c)$

when $M \leq f^{(n+1)}(x)$

for $x \in (b,c)$

$\left.\begin{array}{l} \sin(.5) \\ \cos(2.3) \end{array}\right\}$ $f^{n+1}(x) = \begin{cases} \sin(\theta) \to \\ \cos(\theta) \to \end{cases}$ both ≤ 1 everywhere

$\ln(x)$ $\}$ $f^{n+1}(x) = \left\{ \dfrac{n!}{x^{n-1}} \right.$ not bounded on any interval containing 0

choose interval that
1. contains your input (.5 or 2.3)
2. $f^{n+1} \leq M$ on that interval

Approximate $f(x) = 3\cos x + 1$ @ $x = 0.3$, up to 10^{-3}

• $a = 0$, since $0.3 \approx 0$

$3\cos(0.3) + 1$

$$|R_n(x)| \leq \frac{M \cdot |x|^{n+1}}{(n+1)!} = \frac{3|x|^{n+1}}{(n+1)!} < 10^{-3}$$

sub $x = 0.3$ & solve

what is M?

$f' = -3\sin x$ $\quad -1 \leq \sin x \leq 1$ $\quad |f'| < 3$

$f'' = -3\cos$ $\quad |f''| < 3$

$\vdots$

$|f^{(n+1)}| = \begin{cases} |\pm 3\cos x| \\ |\pm 3\sin| \end{cases}$ both ≤ 3

$$\frac{3 \cdot (0.3)^{n+1}}{(n+1)!} < 10^{-3}$$

$$10^3 < \frac{(n+1)!}{3(0.3)^{n+1}} = \frac{(n+1)! \left(\frac{10}{3}\right)^{n+1}}{3}$$

$n = 4$

$$\approx 5 \cdot \frac{(3)^5}{3} = \frac{5}{3} \cdot 3^5 = \frac{5}{3} 2.43$$

$n = 5$ ☺

$$\cos(x) = \sum (-1)^n \frac{x^{2n}}{(2n)!}$$

$3\cos(x) + 1 = 3\sum M + 1$

$$3\cos(0.3) + 1 \approx 3\left(1 - \frac{0.3^2}{2!} + \frac{0.3^4}{4!} - \frac{0.3^6}{6!} + \frac{0.3^8}{8!} - \frac{0.3^{10}}{10!}\right)$$