Approximate $\ln(1.3)$ to within 10^{-3}.

1. Build Taylor Series @ $a=1$

2. $f(x) = \ln x \qquad f(1) = 0 \qquad - \quad 0$

$f' = \frac{1}{x} \qquad f'(1) = 1 \qquad \frac{1}{1!} = 1 \qquad \ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$

$f'' = -\frac{1}{x^2} \qquad f''(1) = -1 \qquad \frac{-1}{2!} = -\frac{1}{2} = \frac{-1}{2} \qquad \ln(1) = 0$ as known

$f''' = \frac{2}{x^3} \qquad f''' = 2 \qquad \frac{2}{3!} = \frac{2!}{3} = \frac{1}{3}$ T.R.Thm $\Rightarrow$

$|R_n(x)| \le \frac{M \cdot |x-1|^{n+1}}{(n+1)!}$

$f'''' = \frac{-6}{x^4} \qquad f'''' = -6 \qquad \frac{-6}{4!} = -\frac{3!}{4!} = \frac{-1}{4}$

$\frac{3}{10} \quad \frac{9}{100}$

$$\boxed{\begin{aligned} \ln(1.3) &\approx .3 + \frac{(.3)^2}{2} = 0.7 + \frac{0.09}{2} \\ &\approx \end{aligned}}$$

$$|R_n(x)| \le \frac{M \cdot |x-1|^{n+1}}{(n+1)!} < 10^{-3}$$

$f^{(n+1)}(x) = \frac{n!}{x^{n+1}} \quad \Big| \quad f^{(n+1)}(0.5) = \frac{n!}{(\frac{1}{2})^{n+1}} = \frac{n!}{2^{n+1}} \quad \Big| \quad \frac{n! \cdot |x-1|^{n+1}}{2^{n+1}(n+1)!} = \left(\frac{x-1}{2}\right)^{n+1} \cdot \frac{1}{n}$

If we chose $c = 0.5$

the $(c, 100)$ contains 1 & the $f^{n+1}(x)$ will be maximized there.

$\left(\frac{x-1}{2}\right)^{n+1} \cdot \frac{1}{n} < 10^{-3} \Rightarrow 1000 < \frac{n2^{n+1}}{(1.3-1)^{n+1}} = \frac{n2^{n+1}}{(0.3)^{n+1}} = n\left(\frac{2}{0.3}\right)^{n+1} = n\left(\frac{20}{3}\right)^{n+1}$

$1000 < n\left(\frac{20}{3}\right)^{n+1} < n(7)^{n+1}$ so

$n=2 \Rightarrow 2 \cdot 7^3 \approx 2 \cdot \underbrace{50 \cdot 7}_{350} \approx 700$

$n=3 \Rightarrow 3 \cdot 7^4 \approx 3 \cdot \underbrace{50 \cdot 50}_{2500} \approx 7500$

$n=3$

chose $c=1 \Rightarrow$

$f^{n+1}(1) = n!$

$|R_n(1.3)| \le \frac{n! \, |1.3-1|^{n+1}}{(n+1)!} = \frac{n! (0.3)^{n+1}}{(n+1)!} = \frac{(0.3)^{n+1}}{n+1} < 10^{-3} \Rightarrow 1000 < \frac{n+1}{(0.3)^{n+1}} = \left(\frac{10}{3}\right)^{n+1} \cdot (n+1)$

$n=4 \Rightarrow 3^5 (6) = 81 \cdot 3 \cdot 6 = 246 \cdot 6 > 1000$

$\left(\frac{10}{3}\right)^{n+1}(n+1) \approx 3^{n+1}(n+1) \qquad n=4$

this made it close!

$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{4}$

$\ln(1.3) = 0.3 - \frac{(0.3)^2}{2} + \frac{0.3^3}{3} - \frac{0.3^4}{4}$

Ex. Approximate $\ln(1.3)$ up to 10^{-3} accuracy.

① use a Taylor series: choose "a" close to 1.3, $\boxed{\text{so } a=1}$

You could use 0, the only difference in the end will be the number of terms needed. "a closer to 1.3, means less 'n' "

② Taylor's Rem. Thm: $\underbrace{|R_n(x)|}_{\text{error @ } x} \le \dfrac{M \cdot |x-a|^{n+1}}{(n+1)!}$ where $M \geq f^{(n+1)}(x)$ for

③ Find M:

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

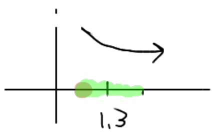
$f'' = \frac{-1}{x^2}$, $1 = (2-1)!$

$f''' = \frac{2}{x^3}$, $2 = (3-1)!$

$f'''' = \frac{-6}{x^4}$, $6 = (4-1)!$

Pattern: $f^{(n+1)}(x) = (-1)^{n+1} \dfrac{n!}{x^{n+1}}$

· Restrict interval so that this ↑ is bounded: i.e., $[1, \infty)$



Fact: $f^{n+1}(x) \le (-1)^{n+1} \dfrac{n!}{(1)^{n+1}}$ on this region.

so $M = n!$

what n makes this true ?

④ $|R_n(1.3)| \le \dfrac{n! \, |1.3-1|^{n+1}}{(n+1)!} = \dfrac{(0.3)^{n+1}}{n+1} < 10^{-3} \Rightarrow 1000 < \dfrac{n+1}{(0.3)^{n+1}} = (n+1)\left(\dfrac{10}{3}\right)^{n+1}$

$(0.3)^{n+1} = \left(\frac{3}{10}\right)^{n+1}$, $(n+1)\left(\frac{10}{3}\right)^{n+1} \geq (n+1) \cdot 3^{n+1}$

$\sum \dfrac{f^n(a)\,|x-a|^n}{n!}$

⑤ $\ln(1.3) = \underbrace{\ln(1)}_{n=0} + \underbrace{\dfrac{1 \cdot (0.3)^1}{1}}_{n=1} - \underbrace{\dfrac{1}{1^2}\dfrac{(0.3)^2}{2!}}_{n=2} + \dfrac{2 \cdot (0.3)^3}{3!} - \dfrac{6(0.3)^4}{4!} + \dfrac{24(0.3)^5}{5!}$

$n=4 \Rightarrow 1000 < 5 \cdot 3^5 = 5 \cdot \underbrace{3^4 \cdot 3}_{243} > 1000$

$=$

n=5 will work

Math 163 - Calculus - Exam 3 - Guide **Name:** ____________________

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Show your work to receive full credit.

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1. Find the third degree Taylor polynomial for $x^{3/2}$ about $x = 1$.

2. Find the arc length of the curve

$$y = \frac{2}{3}x^{3/2} \text{ from } 0 \text{ to } 3$$

3. The Maclaurin series for $\cos x$ is below.

$$\cos x = \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x$$

Find the interval of convergence.

4. Use a series to approximate $\sin\left(\overset{.5}{\cancel{x}}\right)$ to within 10^{-3} accuracy.

5. Use an eighth degree Taylor polynomial to estimate

$$\int_0^1 \frac{\cos(x^2)-1}{x}\,dx =$$

x^8

$$\cos(x) = \sum (-1)^n \frac{x^{2n}}{(2n)!}$$

MaLaurin

$$= 1 - x^4/2 + x^8/24$$

$$\int_0^1 \frac{\text{poly} - 1}{x} = \int_0^1 \text{other poly}\,dx = \frac{x^{n+1}}{n+1} + \dots$$

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

$$\cos(x^2) \approx \text{poly} = \sum(-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{2} (-1)^n \frac{x^{4n}}{(2n)!}$$

$n=0$

$$\frac{x^{4\cdot 0}}{(2\cdot 0)!} - \frac{x^{4\cdot 1}}{(2\cdot 1)!} + \frac{x^{4\cdot 2}}{(2\cdot 2)!} = 1 - \frac{x^4}{2} + \frac{x^8}{24}$$

$$\int_0^1 \frac{1 - \frac{x^4}{2} + \frac{x^8}{24}}{x}\,dx = \int_0^1 -\frac{x^3}{2} + \frac{x^7}{24}\,dx = -\frac{x^4}{8} + \frac{x^8}{8\cdot 24}\Big|_0^1 = \frac{-1}{2\cdot 8}\cdot 24 + \frac{1}{192} - \emptyset$$

$$\boxed{-\frac{23}{192}}$$

6. The Maclaruin series for the funciton ln$(x+1)$ is below. Find the interval of convergence.

$$\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$$

only when

$x \in (-1, 1]$

Ex

ln(2)

$$\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum \frac{(-1)^{n+1} 1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

sub $x=1$ above

7. Use the expansion above to find a series that converges to ln 2.

8. Prove the following (most beautiful) equation is true:

$$e^{i\pi} - 1 = 0.$$

Hint: Find Maclaurin series for $e^x, \cos(x)$ and $\sin(x)$. Then evaluate the series for e^x at $x = i\theta$. Finally, evaluate this series at $\theta = \pi$.

i $i^n =$

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \qquad e^{i\theta} = \sum \frac{(i\theta)^n}{n!} \quad \sum \frac{i^n \theta^n}{n!}$$

$\boxed{\begin{array}{l} n \text{ odd} \Rightarrow \pm i \\ n \text{ even} \Rightarrow \pm 1 \end{array}}$

$$\sin x = \sum (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$$

$$= \sum \frac{(-1)^n \theta^{2n}}{(2n)!} + \frac{(-1)^n i\, \theta^{(2n+1)}}{(2n+1)}$$

$$= \cos\theta + i\sin\theta$$

$$\theta = \pi \Rightarrow \quad e^{i\pi} = 1 + 0$$

$$e^{i\pi} - 1 = 0$$