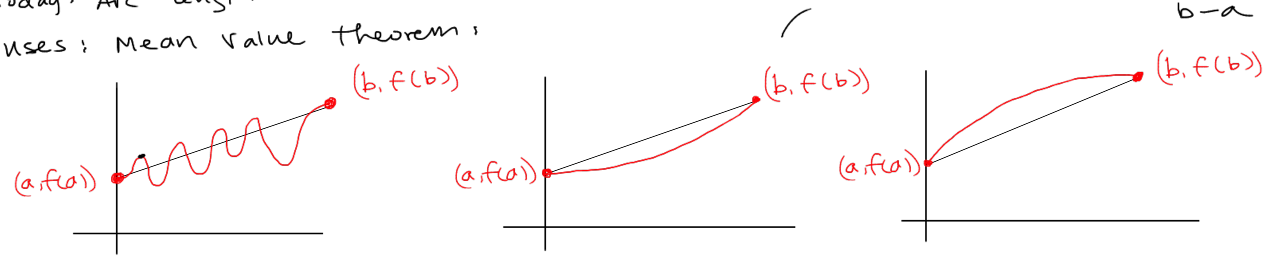
MA163 WK 10

Today: Arc length

uses: Mean value theorem:

slope of secant $\dfrac{f(b)-f(a)}{b-a}$

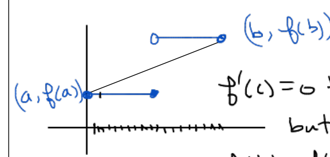


If $f(x)$ is continuous, $\exists$ some $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

The average rate of change is achieved,
as the instantaneous rate of change at some
guaranteed point

vs. $f(x)$ not continuous



$f'(c) = 0 \;\forall\, c$

but

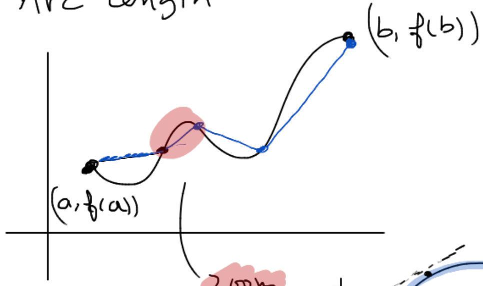
$\dfrac{f(b)-f(a)}{b-a} > 0$

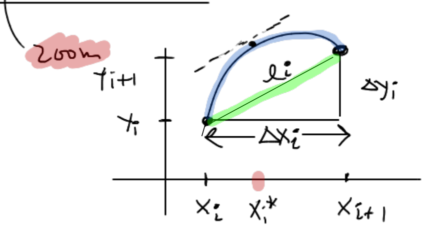
Arc Length

What is the length of this curve?

Use: "calculus" to compute it.

1. approximate it's length by breaking it up into straight sections, computing the length of each, adding up at end





Length of hypotenuse: (Pyth thm)

$$l_i = \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}$$

$$l_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\text{Total Length} \approx \sum_{i=0}^{N} l_i$$

$$l_i = \sqrt{(\Delta x_i)^2\left(1 + \frac{(\Delta y_i)^2}{(\Delta x_i)^2}\right)}$$

Factor out $(\Delta x_i)^2$

$$l_i = \Delta x_i \sqrt{\left(1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right.} = \sqrt{\left(1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right.} \cdot \Delta x_i$$

this quantity is a slope... avg. rate of change

M.V.T. $\Rightarrow \exists$ some $x_i^* \in (x_i, x_{i+1})$ s.t.

$$\frac{\Delta y_i}{\Delta x_i} = f'(x_i^*)$$

$$\text{Total length} \approx \sum_{i=0}^{N} \sqrt{\left(1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)\right)^2} \cdot \Delta x_i = \sum_{i=0}^{N} \sqrt{\left(1 + (f'(x_i^*))\right)^2} \cdot \Delta x_i$$

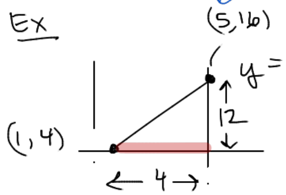
$$\underline{\text{Total Length}} = \underbrace{\lim_{N\to\infty} \sum_{i=0}^{N}}_{} \sqrt{1 + (\underset{\downarrow\, x}{f'(x_i^*)})^2} \cdot \underset{\searrow dx}{\Delta x_i} = \boxed{\int_a^b \sqrt{1 + (f'(x))^2}\, dx \quad \text{Arc length Formula}}$$

as $N \to \infty$

$$\int_a^b f(x)\,dx \equiv \lim_{N\to\infty} \sum_{i=0}^{N} f(x_i^*)\,\Delta x_i$$

length $= \int_a^b \sqrt{1+(f'(x)}\,dx$

Ex



$y = 3x+1 = f(x)$

$f'(x) = 3$

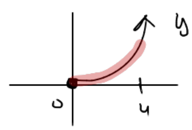
$l = \int_1^5 \sqrt{1+(3)^2}\,dx = \sqrt{10}\int_1^5 dx = \sqrt{10}\,x\Big|_1 = 5\sqrt{10} - 1\sqrt{10}$

$= 4\sqrt{10}$

check using Pyth. Thm

$\sqrt{(12)^2 + (4)^2}$

Ex



$y = x^2$

$l = \int_0^4 \sqrt{1+(2x)^2}\,dx = \int_0^4 \sqrt{1+4x^2}\,dx \approx 16.8$