Thursday – Week 10

Exam Next Week: Thurs / Fri.
Study guide – tonight

Binomial Series.

If r is a positive integer we have a (polynomial) formula for this
$(1+x)^r$

$r=1$ $\quad 1+x$

$r=2$ $\quad 1+2x+x^2$

$r=3$ $\quad 1+3x+3x^2+x^3$

$r=4$ $\quad 1+4x+6x^2+4x^3+x^4$

$\vdots$

$r=n$ $\quad \sum_{k=0}^{r} \binom{r}{k} x^k$

$$\boxed{\binom{r}{k} = \frac{r!}{k!(r-k)!}}$$

r choose k.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

If r is not a positive integer we still get a polynomial (infinite) like this.

Consider $\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)x^n}{n!}$

Find the MacLaurin Series: (see WeBWorK #10)

$f(x) = (1+x)^r$ $\qquad f(0) = 1$

$f'(x) = r(1+x)^{r-1}$ $\qquad f'(0) = r$

$f''(x) = r(r-1)(1+x)^{r-2}$ $\qquad f''(0) = r(r-1)$

$f'''(x) = r(r-1)(r-2)(1+x)^{r-3}$ $\qquad f'''(0) = r(r-1)(r-2)$

$\vdots$

$f^{(n)}(x) = r(r-1)(r-2)\cdot\ldots\cdot\underbrace{(r-(n-1))}_{r-n+1}(1+x)^{r-n}$ $\qquad \boxed{f^{(n)}(0) = r(r-1)(r-2)\cdot\ldots\cdot(r-n+1)}$

$$(1+x)^r = \sum_{n=0}^{\infty} \frac{r(r-1)(r-2)\cdot\ldots\cdot(r-n+1)\cdot x^n}{n!}$$

define $\binom{r}{n} = \frac{r(r-1)(r-2)\cdot\ldots\cdot(r-n+1)}{n!}$

$$\boxed{(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n}$$ — Just like For Finite

To find the interval of convergence — ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\binom{r}{n+1}x^{n+1}}{\binom{r}{n}x^n}\right| = \left|\binom{r}{n+1}X^{n+1} \cdot \frac{1}{\binom{r}{n}x^n}\right| = \left|\frac{X\cdot(n-r)}{(n+1)}\right| \xrightarrow[n\to\infty]{} |x| < 1$$

Convergence if $\downarrow$

Note: $$\frac{\binom{r}{n+1}}{\binom{r}{n}} = \frac{\dfrac{r(r-1)(r-2)\cdot\ldots\cdot(\overbrace{r-(n+1)+1}^{r-n})}{(n+1)!}}{\dfrac{r(r-1)(r-2)\cdot\ldots\cdot(r-n+1)}{n!}} = \frac{r(r-1)\cdot\ldots\cdot(r-(n-1))(r-n)}{(n+1)!}\cdot\frac{n!}{r(r-1)\cdot\ldots\cdot(r-(n-1))}$$

$$= \frac{(r-n)}{(n+1)!}\cdot n! = \left(\frac{r-n}{n+1}\right)$$

Int. of Conv $= (-1,1)$

Endpoints?

- $r \geq 0$ both endpoints give convergence
- $-1 < r < 0$ convergence @ $x=1$ / divergence for $x=-1$
- $r < -1$ divergence at both endpoints

EX

Give a power series for $f(x)$ of form:

$$\sum_{n=0}^{+\infty} c_n (x-5)^n$$

Taylor Series at $a=5$

$f(x) = x^{.8}$

$f'(x) = .8x^{.8-1}$

$f''(x) = .8(.8-1)x^{.8-2}$

$\vdots$

$$\boxed{f^{(n)}(x) = .8(.8-1)\cdot\ldots\cdot(.8-n+1)\cancel{x}^{.8-n}}$$

$x=5$

$f(5) = 5^{.8}$

$f'(5) = .8\cdot 5^{.8-1}$

$f''(5) = .8(.8-1)\cdot 5^{.8-2}$

$\vdots$

$f^{(n)}(5) = .8(.8-1)(.8-2)\cdot\ldots\cdot(.8-n+1)\cdot 5^{.8-n}$

Interval of Convergence

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(5)}{n!}\cdot(x-5)^n$$

Ratio Test:

$$\left|\frac{\frac{f^{(n+1)}(5)}{(n+1)!}\cdot(x-5)^{n+1}}{\frac{f^{(n)}(5)}{n!}\cdot(x-5)^n}\right| = \left|\frac{f^{n+1}(5)(x-5)^{n+1}}{(n+1)!}\cdot\frac{n!}{f^n(5)\cdot(x-5)^n}\right|$$

$$= \left|\frac{\cancel{.8}(\cancel{.8-1})\cdot\ldots\cdot\cancel{(}.8-\underbrace{(n+1)+1}) (x-5)^{n+1}}{(n+1)!}\cdot\frac{n!}{(.8(.8-1)\cdot\ldots\cdot(.8-\cancel{n+1}))(x-5)^n}\right|$$

$$= \left|\frac{(.8-n)(x-5)}{n+1}\right| \xrightarrow[n\to\infty]{} \frac{|x-5|}{5} < 1$$

b/c abs value

$$\left|\frac{n-.8}{n+1}\cdot(x-5)\right|$$

Ratio test:

Convergence if $|x-5|<1$

$-1 < x-5 < 1$

$\underline{4} < x < 6$

domain of $f(x) = x^{.8}$

$(0,\infty)$

$\underline{\underline{x \le 6}}$

Thursday - Week 10

Binomial Series

Revisit binomials:

Below, assume r is a positive integer.

$$(1+x)^r$$

For each r, we get a different polynomial,

$$(1+x) = 1+x$$

$$(1+x)^2 = 1+2x+x^2$$

$$(1+x)^3 = 1+3x+3x^2+x^3$$

$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$

$k \longrightarrow$

1

1 1 $\quad n=1$

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1 $\quad r=5$

$k=0 \quad k=1 \quad k=4$

r

Each number is $\binom{r}{k}$ "r choose k"

$$\boxed{\binom{r}{k} = \frac{r!}{k!(r-k)!}}$$

# of ways to choose k items from a bag of r.

If we let r be anything in $\mathbb{R}$ we get an infinite polynomial via MacLaurin Series

Construct MacLaurin Series

$$f(x) = (1+x)^r \qquad f(0) = 1$$

$$f'(x) = r(1+x)^{r-1} \qquad f'(0) = r$$

$$f''(x) = r\cdot(r-1)(1+x)^{r-2} \qquad f''(0) = r(r-1)$$

$$f'''(x) = r(r-1)(r-2)(1+x)^{r-3} \qquad f'''(0) = r(r-1)(r-2)$$

$$\vdots$$

$$f^n(x) = r(r-1)(r-2)(r-3)\cdot\ldots\cdot(r-(n-1))(1+x)^{r-n}$$

$\updownarrow$ same

$$f^{(n)}(0) = r(r-1)(r-2)(r-3)\cdot\ldots\cdot(r-n+1)$$

$$\text{MacLaurin Series} = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \left[\frac{r(r-1)(r-2)\ldots(r-n+1)}{n!}\right] x^n$$

(define to be $\binom{r}{n}$ agrees with the def'n above when r is an integer.

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

what is the interval of convergence

$$\sum_{n=0}^{\infty} \left[\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}\right] x^n$$

Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\dfrac{r(r-1)(r-2)\cdot\ldots\cdot(\overbrace{r-(n+1)+1}^{r-n-1+1\ =\ r-n})}{(n+1)!}\, x^{n+1}}{\dfrac{r(r-1)(r-2)\cdot\ldots\cdot(r-n+1)}{n!}\cdot x^n} \right|$$

$$= \left| \frac{\cancel{r(r-1)(r-2)\cdot\ldots\cdot(r-n+1)}(r-(n+1)+1)}{(n+1)!} x^{n+1} \cdot \frac{n!}{\cancel{r(r-1)(r-2)\cdot\ldots\cdot(r-n+1)}\, x^n} \right|$$

$$= \left| \frac{(r-n)\,x}{(n+1)} \right| = \left| \frac{(n-r)}{(n+1)} \cdot x \right| \xrightarrow[n\to\infty]{} |x|$$

Ratio Test convergence when $|x| < 1$

$\boxed{x \in (-1,1) \text{ Interval of Conv}}$

Endpoints depends on r — see $\boxed{\#10 \text{ on WW}}$

if $r \geq 0$ endpoints give convergence | series converges in $[-1,1]$

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if $-1 \leq r < 0$ converges @ $x=1$ / diverges @ $x=-1$ $(-1,1]$

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if $r < -1$ diverges @ endpoints $(-1,1)$

Ex Give a power series for

$f(x) = x^{.8}$ of form $\sum_{n=0}^{\infty} c_n (x-5)^n$

Find Taylor series centered at 5.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!} \cdot (x-5)^n$$

$f(x) = x^{.8}$

$f'(x) = .8x^{.8-1}$

$f''(x) = .8(.8-1)x^{.8-2}$

$f'''(x) = .8(.8-1)(.8-2)x^{.8-3}$

$\vdots$

$f^{(n)}(x) = .8(.8-1)(.8-2)\cdot\ldots\cdot(.8-(n-1))\, x^{.8-n}$

$f^{(n)}(5) = .8(.8-1)(.8-2)\cdot\ldots\cdot(.8-(n-1))\cdot 5^{.8-n}$

$f(5) = 5^{.8}$

$f'(5) = .8(5)^{.8-1}$

$f''(5) = .8(.8-1)(5)^{.8-2}$

$f'''(5) = .8(.8-1)(.8-2)(5)^{.8-3}$

$\vdots$

Determine the int. of conv for $\sum \frac{f^{(n)}(5)}{n!}(x-5)^n$

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| =$

$$\frac{\cancel{.8(.8-1)\cdot\ldots\cdot(.8-n+1)}(\overbrace{.8-(n+1)+1}^{.8-n})5^{.8-(n+1)} \cdot (x-5)^{n+1}}{(n+1)!} \cdot \frac{n!}{\cancel{.8(.8-1)(.8-2)\cdot\ldots\cdot(.8-(n-1))}\cdot 5^{.8-n}(x-5)^n}$$

$$\left| \frac{(.8-n)5^{.8-(n+1)}\cdot(x-5)^{n+1}}{(n+1)\, 5^{.8-n}(x-5)^n} \right|$$

top $5^{.8-n-1} = 5^{.8-n}\cdot 5^{-1}$

$$\left|\frac{(.8-n)5^{-1}\cdot(x-5)}{n+1}\right| \xrightarrow[n\to\infty]{} \left|\frac{x-5}{5}\right| < 1$$

Conv.

$-5 < x-5 < 5$

$0 < x < 10$

$x \in (0,10)$

$r = .8 > 0$

$[0,10]$