warm-up – Wed — Week 10 — #10 WW

① MacLaurin Series for e^x
② when does it converge

MacLaurin = Taylor at $a=0$

$$\left[= \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \right] \quad \text{if } a=0 \text{ then}$$

$$= \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)\,x^n}{n!}$$

Compute the red coefficients:

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$\vdots$

$$\boxed{e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}}$$

So

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Note $\quad \downarrow d/dx$

$$e^x = 1 + x + x^2 + \ldots$$

Interval of Convergence

Ratio Test: $\left| \frac{\cancel{x}^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

algebra $= \left| \frac{x}{n+1} \right| \xrightarrow[n \to \infty]{} 0$

$\Rightarrow$ no conditions on x

$\Rightarrow$ Int. of Conv $= (-\infty, \infty)$

Two Follow-up questions to

$$\boxed{e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}}$$ ⟵ works for any $x \in \mathbb{R}$

① Give a series that converges to e.

sub $x=1$.

$$e^1 = \sum_{n=0}^{+\infty} \frac{1^n}{n!}$$

Fact:	
$0! \equiv 1$ b/c $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3!$	$\overset{1}{\overset{\shortparallel}{1!}} = 1 \cdot 0!$

$$e = \sum_{n=0}^{+\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$$

② Give a power series for $f(x) = e^{-x^2}$

sub $-x^2$ in for x

Note: $(x^2)^n = x^{2n}$

$$e^{-x^2} = \sum_{n=0}^{+\infty} \frac{(-x^2)^n}{n!} = \boxed{\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{n!}}$$

This is helpful b/c this is impossible without:

$$\int_0^1 e^{-x^2}\,dx$$ ⟵ We approximate this by polynomial and integrate:

Degree 4 approximation:

$$\underset{n=0}{1} - \underset{n=1}{\frac{1x^2}{1!}} + \underset{n=2}{\frac{x^4}{2!}} = 1 - x^2 + \frac{x^4}{2} \approx e^{-x^2}$$

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 1 - x^2 + \frac{x^4}{2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{10}\Big|_0^1 = 1 - \frac{1}{3} + \frac{1}{10}$$

$$= \frac{30 - 10 + 3}{30}$$

$$= \frac{23}{30}$$

Note:

$$\int_1^3 e^{-x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{10}\Big|_1^3 = 3 - 9 + 24.3 - \frac{23}{30}$$