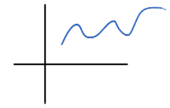
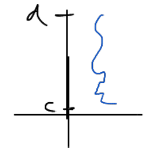
MA162 Wk 10 Wed

Arc Length:

For a curve $y = f(x)$ on $x \in (a,b)$ its length $= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

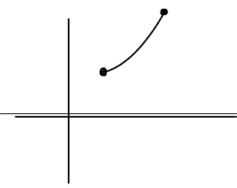


or $(x = f(y))$ on $y \in (c,d)$ " $= \int_c^d \sqrt{1 + (f'(y))^2}\, dy$



Ex. Sometimes the integrals arising from arc length calculations aren't too bad

$y^2 = x^3$ determines a curve b/w $(1,1)$ & $(4,8)$



$y = \sqrt{x^3} \rightarrow \boxed{\frac{dy}{dx} = \frac{3}{2}x^{1/2}}$

$y = x^{3/2}$

or: implicit diff

$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3)$

$2y \cdot \frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{2y} = \frac{3x^2}{2x^{3/2}} = \frac{3}{2}x^{1/2}$

$\int_1^4 \sqrt{1 + (\frac{3}{2}x^{1/2})^2}\, dx$

"

$\int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$

$\boxed{u = 1 + \frac{9}{4}x}$

$du = \frac{9}{4}dx$

$\frac{4}{9}du = dx$

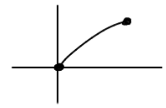
when $x = 1 \Rightarrow u = 1 + \frac{9}{4} = \frac{13}{4} = 3.25$

$x = 4 \Rightarrow u = 1 + \frac{9}{4}(4) = 10$

"

$\int_{3.25}^{10} \sqrt{u} \cdot \frac{4}{9}du = \frac{4}{9}\int_{3.25}^{10} u^{1/2} du = \frac{4}{9} \cdot \frac{2}{3} u^{3/2}\Big|_{3.25}^{10} = \frac{4}{9} \cdot \frac{2}{3}\left[10^{3/2} - 3.25^{3/2}\right] =$

Ex. Often the integrals that appear in arc length calculations are either trig int or trig sub



$\frac{1}{\cos(\pi/4)} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

Curve: $y = \ln(\sec(x))$ b/w $x = 0$ & $x = \frac{\pi}{4}$

$y' = \frac{1}{\sec(x)} \cdot \sec(x)\tan(x) = \tan x$

$$\ell = \int_0^{\pi/4} \sqrt{1 + (\tan(x))^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx$$

$\frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2}$

$$= \int_0^{\pi/4} |\sec(x)|\, dx = \int_0^{\pi/4} \sec(x)\, dx = \int_0^{\pi/4} \sec(x) \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$$

positive x's

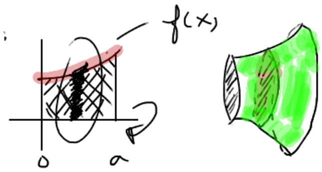
$u = \sec x + \tan x$
$du = \sec^2 x + \sec x \tan x$
$x = 0 \Rightarrow u = 1 + 0 = 1$
$x = \frac{\pi}{4} \Rightarrow u = \sqrt{2} + 1$

$$= \int_0^{\pi/4} \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\, dx = \int_1^{\sqrt{2}+1} \frac{du}{u}$$

$$= \ln|u| \Big|_1^{\sqrt{2}+1} = \ln(\sqrt{2}+1) - \overset{=0}{\ln(1)} = \ln(\sqrt{2}+1)$$

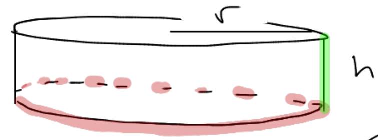
Surface Area

Recall:



$$V = \int_0^a \pi (f(x))^2 \, dx$$

volume

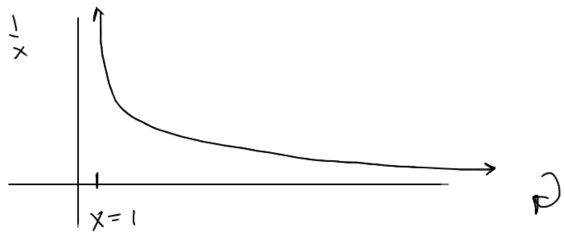


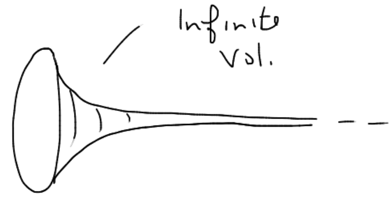
$$A = 2\pi r \overset{\text{length}}{h}$$

$$S = \int_0^a 2\pi \cdot \overset{\text{radius}}{f(x)} \cdot \underbrace{\sqrt{1 + (f'(x))^2}}_{\text{length} \Leftrightarrow h} \, dx$$

Gabriel's Horn & Painter's Paradox

$f(x)=\frac{1}{x}$





Recall $V=\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \int_1^\infty \pi x^{-2} = \pi\frac{x^{-1}}{-1}\Big|_1^\infty = \frac{-\pi}{x}\Big|_1^\infty = \left(\frac{-\pi}{\infty} - \frac{-\pi}{1}\right)$

$\to 0 + \pi$ Finite!

$\left(\frac{1}{x}\right)'$

$$S=\int_0^\infty 2\pi\cdot\left(\frac{1}{x}\right)\underbrace{\sqrt{1+\left(\frac{-1}{x^2}\right)^2}}\,dx = \underbrace{\int_1^\infty 2\pi\frac{\sqrt{x^4+1}}{x^3}\,dx}_{\text{goal}} > \int_1^\infty 2\pi\frac{\sqrt{x^4}}{x^3}\,dx$$

area

$\to \sqrt{1+\frac{1}{x^4}}$

$\to \sqrt{\frac{x^4+1}{x^4}}$

$\to \frac{\sqrt{x^4+1}}{\sqrt{x^4}}$

$\frac{\sqrt{x^4+1}}{x^2}$

goal $\to \infty$

$2\pi\int_1^\infty \frac{x^2}{x^3}dx = 2\pi\int_0^\infty \frac{1}{x}dx$

$= 2\pi \ln|x|\,\Big|_0^\infty$

$= 2\pi(\ln\infty - \ln 1)$

$= \infty$