1. Find the arc length . . . .
(a)

$$x = \frac{y^4}{8} + \frac{1}{4y^2}, 1 \le y \le 2$$

$$\int_1^2 \sqrt{(1 + (\frac{y^3}{2} - \frac{y^{-3}}{2})^2}\, dy$$

$$= \int_1^2 \sqrt{1 + \frac{1}{4}(y^3 - y^{-3})^2}$$

$$= \int_1^2 \sqrt{1 + \frac{y^6}{4} - \frac{1}{2} + \frac{y^{-6}}{4}}$$

$$= \int_1^2 \sqrt{\frac{y^6}{4} + \frac{1}{2} + \frac{y^{-6}}{4}}\, dy$$

$$= \int_1^2 \sqrt{\frac{1}{4}(y^6 + 2 + y^{-6})}\, dy = \frac{1}{2}\int_1^2 y^3 + y^{-3}\, dy = \frac{1}{2}\left(\frac{y^4}{4} - \frac{y^{-2}}{2}\right)\Big|_1^2 = \frac{1}{2}\left(\overset{=4}{\frac{16}{4}} - \frac{1}{8} - \frac{1}{4} + \frac{1}{2}\right)$$

$$\sqrt{(y^3 + y^{-3})^2} = \sqrt{y^6 + 2 + y^{-6}}$$

$$= \frac{1}{2}\left(\frac{y^4}{4} - \frac{1}{2y^2}\right)\Big|_1^2 = \frac{1}{2}\left[\frac{16}{4} - \frac{1}{8} - \frac{1}{4} + \frac{1}{2}\right.$$

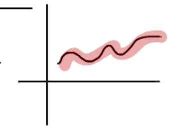
$$= 2 - \frac{1}{16} - \frac{1}{8} + \frac{1}{4}$$

$$= 2 - \frac{1}{16} - \frac{2}{16} + \frac{4}{16}$$

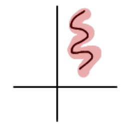
$$= 2\frac{1}{16} = \frac{33}{16}$$

Arc length

For a curve $y = f(x)$ on (a,b) the length $= \ell = \int_a^b \sqrt{1+(f'(x))^2}\,dx$



or $(x = g(y))$ on (c,d) $\ell = \int_c^d \sqrt{1+(g'(y))^2}\,dy$



Ex. Sometimes the integrals are easy

$y^2 = x^3$ determines a curve $(1,1)$ & $(4,8)$, determine its length

① $y = \sqrt{x^3}$ (keep + root since y's here are +.

$= x^{3/2}$

② $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$

③ $\ell = \int_1^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_1^4 \sqrt{1+\frac{9}{4}x}\,dx$

$\boxed{u = 1+\frac{9}{4}x}$ ☆

$du = \frac{9}{4}dx$

$\boxed{\frac{4}{9}du = dx}$

when $x=1$, $u = 1+\frac{9}{4}(1)$

$= 13/4$

$x=4$ $u = 1+\frac{9}{4}(4)$

$= 10$

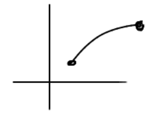
④ $\ell = \int \sqrt{u}\cdot\frac{4}{9}du = \frac{4}{9}\int_{3.25}^{10} u^{1/2}\,du = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\Big|_{3.25}^{10}\right]$

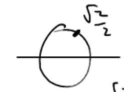
⑤ $\frac{4}{9}\left[\frac{2}{3}\left[10^{3/2} - (3.25)^{3/2}\right]\right] \approx 7.6$

Ex. Often, the integrals obtained from arc length calculations require a trig sub / trig integral

Find length of curve $y=\ln(\sec(x))$ b/w $x=0$, $x=\pi/4$

$$y' = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x)$$





$$\sqrt{2} = \frac{2}{\sqrt2}\cdot\frac{\sqrt2}{\sqrt2}$$

$$(\ln u)' = \frac{du}{u}$$

$$\ell = \int_0^{\pi/4}\sqrt{1+(\tan(x))^2}\,dx$$

$$= \int_0^{\pi/4}\sqrt{\sec^2(x)}\,dx = \int_0^{\pi/4}\sec(x)\,dx$$

$$\frac{\sin^2}{\cos^2}+\frac{\cos^2}{\cos^2}=\frac{1}{\cos^2}$$

$$= \int_0^{\pi/4}\sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx = \int_0^{\pi/4}\frac{\sec^2 x+\sec x\tan x}{\sec(x)+\tan(x)}\,dx$$

$u=\sec x+\tan x$

$du = \sec x\tan x+\sec^2 x$

(*) $x=0 \Rightarrow u=\sec 0+\tan 0 = 1+0=1$

(**) $x=\frac{\pi}{4} \Rightarrow u=\sec\frac{\pi}{4}+\tan\frac{\pi}{4} = \sqrt2+1$

$$= \int_1^{\sqrt2+1}\frac{du}{u} = \ln|u|\Big|_1^{\sqrt2+1} = \ln(\sqrt2+1) - \overset{=0}{\ln(1)} = \ln(\sqrt2+1)$$

---

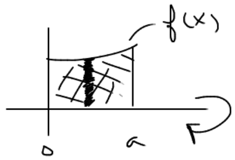
alt.

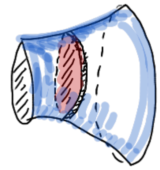
$$\int_\square^\square\frac{du}{u} = \ln|u|\Big|_\square^\square = \ln|\sec x+\tan x|\Big|_0^{\pi/4}$$

$$= \ln\Big|\underbrace{\sec\tfrac{\pi}{4}}_{\sqrt2}+\underbrace{\tan\tfrac{\pi}{4}}_{1}\Big| - \ln\Big|\overset{=1}{\sec(0)}+\overset{=0}{\tan(0)}\Big|$$

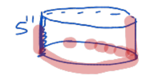
New Topic: Surface Area

Recall: Volume of Solid of Revolution





$$V = \int_0^a \pi \left(f(x)\right)^2 dx$$



$c = 2\pi r$

$r \to 0$

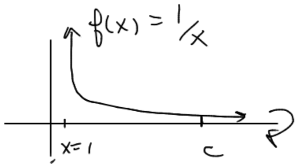
$2\pi r \cdot s$

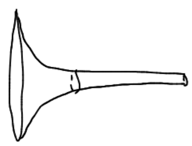
$2\pi r \cdot h$ — arc length

Surface Area:

$$\int_0^a 2\pi y \sqrt{1 + (f'(x))^2}\, dx = \int_0^a 2\pi \cdot f(x) \sqrt{1 + (f'(x))^2}\, dx$$

Gabriel's Horn & Painter's Paradox





Finite Volume
∞ - surface area

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

surface area up to c

$$S = \int_1^c 2\pi \cdot \frac{1}{x} \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx = 2\pi \int_1^c \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx = 2\pi \int_1^c \frac{1}{x} \sqrt{\frac{x^4+1}{x^4}}\, dx$$

$$= 2\pi \int_1^c \frac{\sqrt{x^4+1}}{x^3} dx > 2\pi \int_1^c \frac{\sqrt{x^4}}{x^3} dx = 2\pi \int_1^c \frac{1}{x} = 2\pi \left(\ln|x|\right)\Big|_1^c = 2\pi\left(\ln(c) - \ln(1)\right)$$

$\sqrt{x^4} = x^2$, $\ln(1) = 0$

$$= 2\pi \cdot \ln(c)$$

amount of paint

as $c \to \infty$

$S \to \infty$

$$\lim_{c \to \infty} 2\pi \ln(c) = 2\pi \ln \infty = \infty$$