MA 163 Exam 3

1. Find the third degree Taylor polynomial for $x^{3/2}$ about $x = 1$.

$$f(x) = x^{3/2} \qquad f(1) = 1$$

$$f'(x) = \tfrac{3}{2}x^{1/2} \qquad f'(1) = \tfrac{3}{2}$$

$$f''(x) = \tfrac{3}{4}x^{-1/2} \qquad f''(1) = \tfrac{3}{4}$$

$$f^{(3)}(x) = -\tfrac{3}{8}x^{-3/2} \qquad f^{(3)}(1) = -\tfrac{3}{8}$$

Using the above calculations (and don't forget the factorials) . . . .

$$x^{3/2} \approx 1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{1}{16}(x-1)^3$$

2. Use integrals to estimate the series below to within 0.001 of the actual value.

$$\sum_{n=4}^{+\infty} \frac{1}{4n^3}$$

Since $1/4(7^3) < 0.001$, we can use 7 as the point at which to apply the integral to estimate the tail of the series.

$$\frac{1}{4(4^3)} + \frac{1}{4(4^3)} + \frac{1}{4(6^3)} + \int_7^{+\infty} \frac{1}{4x^3}\,dx \le \sum_{n=4}^{+\infty} \frac{1}{4n^3} \le \frac{1}{4(4^3)} + \frac{1}{4(4^3)} + \frac{1}{4(6^3)} + \frac{1}{4(7^3)} + \int_7^{+\infty} \frac{1}{4x^3}\,dx$$

3. Evaluate the integral:

$$\int_7^{+\infty} \frac{1}{4x^3}\,dx = \lim_{b\to+\infty} \int_7^b \frac{1}{4}x^{-3}\,dx = \lim_{b\to+\infty} \left[-\frac{1}{8}x^{-2}\right]_7^b$$

$$= \lim_{b\to+\infty} \left[-\frac{1}{8b^2} - \left(-\frac{1}{392}\right)\right] = \frac{1}{392}$$

So . . .

$$\frac{1}{4(4^3)} + \frac{1}{4(5^3)} + \frac{1}{4(6^3)} + \frac{1}{392} \le \sum_{n=4}^{+\infty} \frac{1}{4n^3} \le \frac{1}{4(4^3)} + \frac{1}{4(5^3)} + \frac{1}{4(6^3)} + \frac{1}{4(7^3)} + \frac{1}{392}$$

$$0.00961 \le \sum_{n=4}^{+\infty} \frac{1}{4n^3} \le 0.0103$$

4. The Maclaurin series for $\cos x$ is below.

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Find the interval of convergence.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{|x|^{2(n+1)}}{(2(n+1))!}}{\frac{|x|^{2n}}{(2n)!}} = \frac{(2n)!}{(2n+2)!} \cdot \frac{|x|^{2n+2}}{|x|^{2n}}$$

$$= \frac{(2n)!}{(2n+2)(2n+1)(2n)!}|x|^2 = \frac{1}{(2n+2)(2n+1)}|x|^2$$

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{1}{(2n+2)(2n+1)}|x|^2$$

$$= |x|^2 \lim_{n\to+\infty} \frac{1}{(2n+2)(2n+1)} = |x|^2 \cdot 0 = 0 < 1$$

By the Ratio Test, this series converges for all values of x.

5. Use the series above to obtain an estimate for $\cos\left(\frac{1}{2}\right)$ to within 0.01 of the actual vallue.

$$\cos\left(\frac{1}{2}\right) = 1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{24}\left(\frac{1}{2}\right)^4 \simeq 0.8776$$

(Alternating series, and the last fraction is less than 0.01, so we can "stop" here.)

6. Use an eighth degree Taylor polynomial to estimate

$$\int_0^1 \frac{\cos(x^2) - 1}{x}\,dx \simeq \int_0^1 -\frac{1}{2}x^3 + \frac{1}{24}x^7\,dx = \left[-\frac{1}{8}x^4 + \frac{1}{192}x^8\right]_0^1 = -\frac{1}{8} + \frac{1}{192} = -\frac{23}{192}$$

$$\left(\cos(x^2) \simeq 1 - \frac{1}{2}\left(x^2\right)^2 + \frac{1}{24}(x^2)^4 = 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 \Longrightarrow \frac{\cos(x^2) - 1}{x} \simeq -\frac{1}{2}x^3 + \frac{1}{24}x^7\right)$$

7. The Maclaruin series for the funciton $\ln(x+1)$ is below. Find the interval of convergence.

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{(-1)^{n+1+1}}{n+1} x^{n+1}\right|}{\left|\frac{(-1)^n}{n+1} x^n\right|} = \frac{n}{n+1} \cdot \frac{|x|^{n+1}}{|x|^n} = \frac{n}{n+1}|x|$$

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{n}{n+1}|x| = |x| \lim_{n\to+\infty} \frac{n}{n+1} = |x| \cdot 1 = |x|$$

From the Ratio Test, we know that the series converges for $-1 < x < 1$ and diverges for $(-\infty, -1) \cup (1, +\infty)$. We have to check whether or not the series converges for $x = 1$ and $x = -1$.
When $x = 1$, the power series is equal to

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ which converges}$$

When $x = -1$, the power series is equal to

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots \text{ which diverges}$$

Therefore, the interval of convergence for the Maclaurin Series for $\ln(x+1)$ is $(-1, 1]$.

8. Use # 8 to find a series that converges to $\ln 2$.

Plug $x = 1$ into the series in #8: $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

9. Prove $e^{i\pi}+1=0$ We know

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

so

$$e^{i\pi} = \sum_{n=0}^{+\infty} \frac{(i\pi)^n}{n!} = 1 + \frac{i\pi}{1} + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} + \frac{(i\pi)^4}{4!} + \frac{(i\pi)^5}{5!} + \frac{(i\pi)^6}{6!} + \dots$$

since the series converges we can rearrange the order of the terms, keeping the even powers separate from the odd

$$e^{i\pi} = \sum_{n=0}^{+\infty} \frac{(i\pi)^n}{n!} = 1 + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^4}{4!} + \frac{(i\pi)^6}{6!} + \dots + \frac{i\pi}{1} + \frac{(i\pi)^3}{3!} + \frac{(i\pi)^5}{5!} + \dots$$

using the fact that $i^2=-1$ and $i^4=1$ we get

$$e^{i\pi} = \sum_{n=0}^{+\infty} \frac{(i\pi)^n}{n!} = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots + \frac{i\pi}{1} - \frac{i(\pi)^3}{3!} + \frac{i(\pi)^5}{5!} + \dots$$

factoring out i in the second half

$$e^{i\pi} = \sum_{n=0}^{+\infty} \frac{(i\pi)^n}{n!} = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots + i\left(\frac{\pi}{1} - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} + \dots\right)$$

reconizing the odd powers are Maclaurin series for cosine evaluated at $x = i\pi$ and likewise for the odds and sine.

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0 = -1$$

thus

$$e^{i\pi} + 1 = 0$$