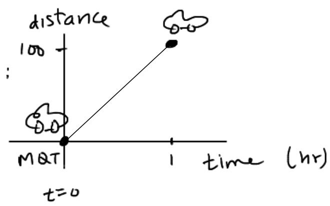
Fn. wk 11

Arc length of a Parametric Curve. → coords are each functions of t

Main Tool:

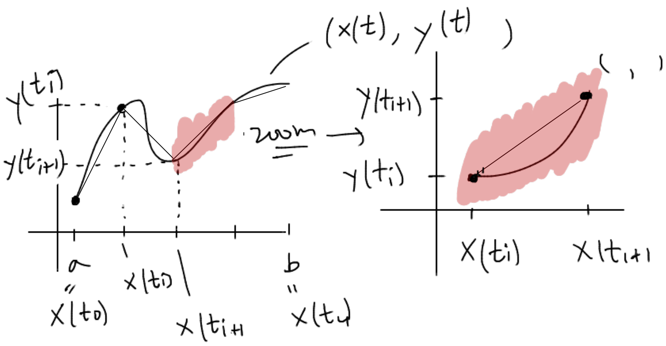
Mean Value Theorem:



Since dist. is a continuous fcn of time, Avg Speed is realized @ some single point in time.

Avg Speed $= 100 \frac{mi}{hr}$ ∃ time t^* s.t. your speedometer (derivative of dist. function) $= 100$

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(t^*)}$$



$$\sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2} = \text{segment length}$$

$$\sqrt{[x'(t_i^*)\cdot \Delta t_i]^2 + [(y'(t_i^*))(\Delta t_i)]^2} =$$

$$\sqrt{[x'(t_i^*)]^2 \cdot \Delta t_i^2 + [y'(t_i^*)]^2 \cdot \Delta t_i^2} = \qquad \sqrt{AB} = \sqrt{A}\sqrt{B}$$

$$\sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2} \; \Delta t_i$$

MVT
$$\frac{x(t_{i+1}) - x(t_i)}{\underbrace{t_{i+1} - t_i}_{\Delta t_i}} = x'(t_i^*)$$

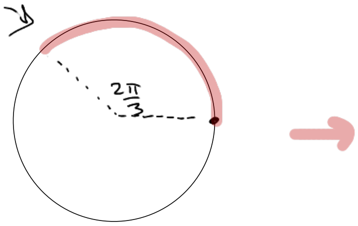
$$\text{Total Length} = \lim_{N\to\infty} \sum_{i=0}^{N} \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2} \; \Delta t_i = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Arc Length Formula for a parametric cuve

Sanity check:

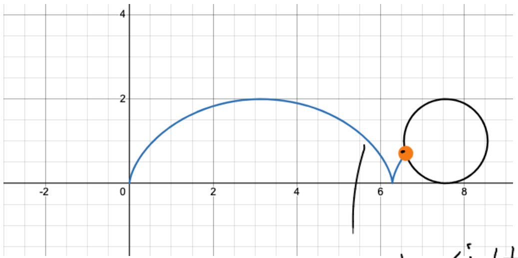
parametrize unit circle

$$\left\{ \begin{array}{l} x = \cos(t) \\ y = \sin(t) \\ 0 \le t \le \frac{2\pi}{3} \end{array} \right\}$$



$$120 \times \frac{\pi}{180} = \frac{2\pi}{3}$$

$$\text{length} = \int_0^{\frac{2\pi}{3}} \sqrt{(x')^2 + (y')^2}\, dt = \int_0^{\frac{2\pi}{3}} \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{=1}}\, dt = \int_0^{2\pi/3} 1\, dt = t\Big|_0^{2\pi/3} = \frac{2\pi}{3}$$



$x = t - \sin(t)$

$y = 1 - \cos(t)$

what's length of curve from $x=0$ to $x=2\pi$?

$$l = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(1-\cos(t))^2 + (\sin(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos(t) + \underbrace{\cos^2(t) + \sin^2(t)}_{=1}}\, dt = \sqrt{2}\int_0^{2\pi} \sqrt{(1-\cos(t))}\, dt$$

$\sqrt{a^2} = a$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2(\tfrac{t}{2})}\, dt = \sqrt{4}\int_0^{2\pi} \sqrt{\sin^2(\tfrac{t}{2})}\, dt$$

HALF-ANGLE

$\sin(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos x}{2}}$

or

$\sin^2(\frac{x}{2}) = \frac{1-\cos(x)}{2}$

$$= 2\int_0^{2\pi} |\sin(\tfrac{t}{2})|\, dt$$

when is $\sin(\frac{t}{2}) < 0$

when $\pi < \frac{t}{2} < 2\pi$

$2\pi < t < 4\pi$

luckily for us $\sin(\frac{x}{2})$ is always positive in our integral

$$= 2\int_0^{2\pi} \sin(\tfrac{t}{2})\, dt$$

$u = \frac{t}{2}$ $\quad t=0 \Rightarrow u=0$

$t = 2\pi \Rightarrow u = \pi$

$du = \frac{1}{2}dt \quad dt = 2du$

e.g. $\int_0^{3\pi} |\sin(\frac{x}{2})| = \int_0^{2\pi} \sin\frac{x}{2} + \int_{2\pi}^{3\pi} -\sin\frac{x}{2}$

$$= 4\int_0^{\pi} \sin(u)\, du$$

$$= 4\left[-\cos(u)\right]\Big|_0^{\pi} = 4\Big(-\underbrace{\cos(\pi)}_{-1} \overset{+}{-} \cos(0)\Big] = 4(-(-1)+1) = 4(1+1) = 8$$

Ellipse!

similar, coef of sin, cos are different

$x = 4\cos t$
$y = \sin t$

$x(t) = h + b\cos t$
$y(t) = k + a\sin t$

circle

$x = \cos t$
$y = \sin t$ } unit circle @ (0,0)

$\downarrow$

$x = a\cos t$
$y = a\sin t$
radius = a circle param.

$x(t) = h + a\cos t$
$y(t) = k + a\sin t$
circle, rad = a
center = h, k.