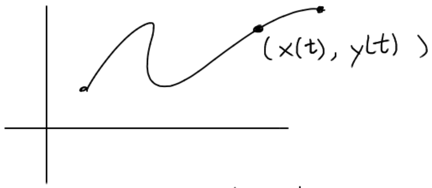
Fri. Wk 11

Arc Length of Parametric Curves:

↳ coordinates are each given by function of a (single) variable

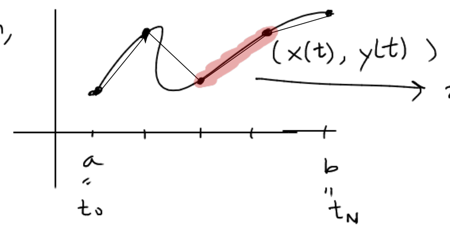


Mean Value Theorem

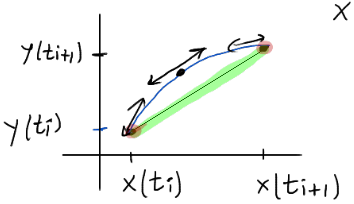
$\exists\ t_i^*$ s.t.

$$\frac{x(t_{i+1}) - x(t_i)}{\underbrace{t_{i+1} - t_i}_{\Delta t_i}} = x'(t_i^*)$$

To find length,



zoom =



single segment length:

$$= \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2}$$

MVT

$$= \sqrt{(x'(t_i^*)\cdot\Delta t_i)^2 + (y'(t_i^*)\cdot\Delta t_i)^2}$$

$$= \sqrt{[x'(t_i^*)]^2\cdot\Delta t_i^2 + [y'(t_i^*)]^2\cdot\Delta t_i^2}$$

$$\sqrt{\left([x'(t_i^*)]^2 + [y'(t_i^*)]^2\right)\cdot\Delta t_i^2}$$

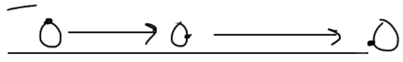
$$= \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\cdot\Delta t_i$$

Total Length: sum over all segment lengths & take limit as # of segments → ∞

$$L = \underbrace{\lim_{N\to\infty} \sum_{i=0}^{N}}\sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\cdot\Delta t_i$$

$$\boxed{L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

Ex



$x(t) = t - \sin(t)$

$y(t) = 1 - \cos(t)$

compute length from $t=0$ to $t = 2\pi$

$$L = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\underbrace{(1-\cos(t))^2 + (\sin(t))^2}}\, dt$$

HALF-ANGLE FORMULAS

$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$

or

$\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{2}$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos(t) + \underbrace{\cos^2(t) + \sin^2(t)}_{=1}}\, dt$$

$$\int_0^{2\pi} \sqrt{2(1-\cos(t))}\, dt = \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos(t)}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\cdot \sin^2\left(\tfrac{t}{2}\right)} = \sqrt{4}\int_0^{2\pi} \sqrt{\sin^2\left(\tfrac{t}{2}\right)}\, dt = 2\int_0^{2\pi} \left|\sin\left(\tfrac{t}{2}\right)\right| dt$$

think when is $\sin\left(\frac{t}{2}\right) < 0$ when $\pi < \frac{t}{2} < 2\pi$ so

$2\pi < t < 4\pi$ Luckily, $\sin(t/2) \geq 0$ on $[0, 2\pi]$

$$= 2\int_0^{2\pi} \sin\left(\tfrac{t}{2}\right) dt = 2\int \sin(u)(2)\,du = 4\int_0^{\pi} \sin(u)\,du = -4\cos(u)\Big|_0^{\pi}$$

$u = t/2$ $\quad$ $t=0 \Rightarrow u=0$ $\quad$ $t=2\pi \Rightarrow u=\pi$

$du = 1/2\, dt \Rightarrow dt = 2\,du$

$-\cos$

$= -4[\cos(\pi) - \cos 0]$

$= -4[(-1) - 1]$

$(-4)(-2) = 8$

Note: Had our problem been:

$$\int_0^{3\pi} |\sin(t/2)|\, dt = \int_0^{2\pi} \sin\left(\tfrac{t}{2}\right) dt + \int_{2\pi}^{3\pi} -\sin\left(\tfrac{t}{2}\right) dt$$

Ellipse!

similar, coef of sin, cos are different

$x = 4\cos t$

$y = \sin t$

$x(t) = h + b\cos t$

$y(t) = k + a\sin t$

circle

$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\}$ unit circle @ (0,0)

$\downarrow$

$x = a\cos t$

$y = a\sin t$

radius = a circle param.

$x(t) = h + a\cos t$

$y(t) = k + a\sin t$

circle, rad = a

center = h, k.