Monday - Week 11

Plan This Week

M: Study Guide

T: Office Hours: 10 am

W: ________ (Office Hours: 8:30 - 9:30 pm)

Th: Exam 3

F: Exam 3

MA 163 Exam 3 Guide

1. Find the third degree Taylor polynomial for $x^{3/2}$ about $x = 1$.

what? $\displaystyle\sum_{n=0}^{3} \frac{f^{(n)}(1)(x-1)^n}{n!}$ where

$f(x) = x^{3/2}$ | $f(1) = 1$

$f'(x) = \frac{3}{2}x^{1/2}$ | $f'(1) = \frac{3}{2}$

$f''(x) = \frac{3}{4}x^{-1/2}$ | $f''(1) = \frac{3}{4}$

$f'''(x) = -\frac{3}{8}x^{-3/2}$ | $f'''(1) = -\frac{3}{8}$

$$= 1 + \frac{3}{2}(x-1) + \frac{3}{2\cdot 4}(x-1)^2 - \frac{3}{3!\cdot 8}(x-1)^3$$

$$1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{3}{48}(x-1)^3$$

2. Use integrals to estimate the series below to within 0.001 of the actual value.

$$\sum_{n=4}^{+\infty} \frac{1}{4n^3}$$

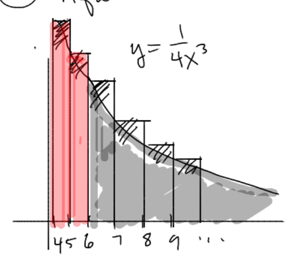
$\frac{1}{4}$, $\frac{1}{4\cdot 8}$, $\frac{1}{4\cdot 3^3}$ | $\frac{1}{4\cdot 4^3}$ $n^3 > 4000$

Idea: By hand find n s.t. $\frac{1}{4n^3} < 0.001$

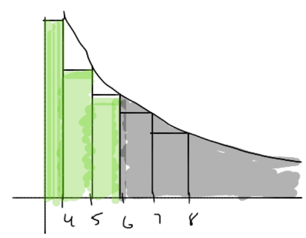
① $n = 6$ works: $\frac{1}{4\cdot 6^3} < 0.001$

② After $n=6$ estimate the tail of the series with an integral.

The Series = Area of Rectangles



$$\frac{1}{4\cdot 4^3} + \frac{1}{4\cdot 5^3} + \int_6^{+\infty} \frac{1}{4x^3}\,dx \le \sum_{n=4}^{+\infty} \frac{1}{4n^3}$$



③ $$\sum_{n=4}^{+\infty} \le \frac{1}{4\cdot 4^3} + \frac{1}{4\cdot 5^3} + \frac{1}{4\cdot 6^3} + \int_6^{+\infty} \frac{1}{4x^3}\,dx$$

$$\frac{1}{4\cdot 4^3} + \frac{1}{4\cdot 5^3} + \int_6^{\infty} \frac{1}{4x^3} \le \sum \frac{1}{4n^3} \le \frac{1}{4\cdot 5^3} + \frac{1}{4\cdot 5^3} + \frac{1}{4\cdot 6^3} + \int_1^{\infty} \frac{1}{4x^3}$$

$\frac{1}{8}$ $\frac{1}{8}$

$$\frac{1}{4\cdot 64} + \frac{1}{4\cdot 125} + \frac{1}{8\cdot 48} \le \sum \frac{1}{4n} \le \frac{1}{4\cdot 64} + \frac{1}{4\cdot 125} + \frac{1}{4\cdot 6^3} + \frac{1}{8\cdot 48}$$

$$\int_6^{\infty} \frac{1}{4x^3}\,dx = -\frac{1}{8}x^{-2}\Big|_6^{\infty}$$

$\frac{1}{4}x^{-3}$

$$\frac{1}{8}\left(\frac{-1}{\infty^2}\right) - \frac{1}{8}\left(\frac{-1}{6^2}\right) = \frac{1}{8\cdot 48}$$

Series

$\Rightarrow$ Between

$(.0085, .0096)$

accurate up to $.001$

3. The Maclaurin series for $\cos x$ is below.

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Find the interval of convergence.

Ratio test $= \left| \frac{a_{n+1}}{a_n} \right| \xrightarrow[n\to\infty]{}$

b/c abs value ignore $(-1)^n$

$$\left| \frac{x^{2(n+1)}}{\underbrace{(2(n+1))!}_{(2n+2)!}} \cdot \frac{(2n)!}{x^{2n}} \right| = \left| \frac{\cancel{x^{2n}} \cdot x^2 \cdot \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\, \cancel{x^{2n}}} \right| = \left| \frac{x^2}{(2n+2)(2n+1)} \right|$$

as $n \to \infty$ this goes to $\underline{\quad 0 \quad}$ which is less than 1

$\Rightarrow$ Int. of Conv. $= (-\infty, \infty)$

Recall
< 1 converge
$= 1$???
> 1 diverge

4. Use the series above to obtain an estimate for $\cos\left(\frac{1}{2}\right)$ to within 0.01 of the actual vallue.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

$$\cos\left(\tfrac{1}{2}\right) = 1 - \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^4}{4!} - \frac{\left(\frac{1}{2}\right)^6}{6!} + \frac{\left(\frac{1}{2}\right)^8}{8!}$$

Alternating Series Test:
An alt. series approximation (truncation) will be accurate up to the absolute value of the next term.

so: $1 - \frac{\left(\frac{1}{2}\right)^2}{2!} = 1 - \frac{1}{4}\left(\frac{1}{2!}\right) = 1 - \frac{1}{8} = \frac{7}{8}$

is within $\left(\text{the next value } \frac{\left(\frac{1}{2}\right)^4}{4!} = \frac{1}{16} \cdot \frac{1}{24}\right)$ of the actual value

Since $\underline{\qquad}$ is less than .01,

$\cos(1/2) \approx \frac{7}{8}$ within .01 accuracy

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$

5. Use an eighth degree Taylor polynomial to estimate

$$\int_0^1 \frac{\cos(x^2) - 1}{x}\, dx =$$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} -$

$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!}$

$$= \int_0^1 \frac{\cancel{1} - \frac{x^4}{2} + \frac{x^8}{4!} - \cancel{1}}{x}\, dx = \int_0^1 \frac{x^3}{2} - \frac{x^7}{24}\, dx$$

$$= \left. \frac{x^4}{8} - \frac{x^8}{8 \cdot 24} \right|_0^1$$

$$= \frac{1}{8} - \frac{1}{8 \cdot 24} = \frac{1}{8}\left(1 - \frac{1}{24}\right)$$

$$= \frac{1}{8}\left(\frac{23}{24}\right) = \frac{23}{8 \cdot 24}$$

6. The Maclaruin series for the funciton $\ln(x+1)$ is below. Find the interval of convergence.

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$$

$(-1, 1]$

Ratio Test: $\left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \left| \frac{x \cdot n}{n+1} \right| \longrightarrow |x|$

(1) converges if $|x| < 1 \Rightarrow x \in (-1,1)$ $\quad -1 < x < 1$

(2) Endpoints: $x = -1 \Rightarrow \frac{(-1)^{n+1} \cdot (-1)^n}{n} = \frac{(-1)^n \cdot (-1) \cdot (-1)^n}{n}$

$\sum_{n=1}^{\infty} -\frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$x = 1 \Rightarrow$

$\sum \frac{(-1)^{n+1} (1)^n}{n} = \sum \frac{(-1)^{n+1}}{n}$ converges by A.S.T

7. Use the expansion above to find a series that converges to $\ln 2$.

sub $x = 1$ $\quad \ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

8. Prove the following (most beautiful) equation is true:

$$e^{i\pi} + 1 = 0.$$

Hint: Find Maclaurin series for $e^x, \cos(x)$ and $\sin(x)$. Then evaluate the series for e^x at $x = i\theta$. Finally, evaluate this series at $\theta = \pi$.

see notes:

$(i)^2 = -1$

Alt. series

$$\sum (-1)^n \cdot b_n$$

If $b_n \longrightarrow 0$

converges.

$$\sum \frac{(-1)^{n+1}}{n} = \sum (-1)^n \cdot \frac{1}{n}$$

$$\frac{1}{n} \longrightarrow 0$$

Monday Week 11

Plan:

M Review of Study Guide

T Office : 10 am

W ——— Office Hours : 8:30 pm

th Exams

F Exams

MA 163 Exam 3 Guide

1. Find the third degree Taylor polynomial for $\boxed{x^{3/2}}$ about $x = 1$.

what! $\sum_{n=0}^{3} \frac{f^{(n)}(1)(x-1)^n}{n!}$ w/ $f(x) = x^{3/2}$

$f(x) = x^{3/2}$ | $f(1) = 1$

$f'(x) = \frac{3}{2}x^{1/2}$ | $f'(1) = \frac{3}{2}$

$f''(x) = \frac{3}{4}x^{-1/2}$ | $f''(1) = \frac{3}{4}$

$f'''(x) = -\frac{3}{8}x^{-3/2}$ | $f'''(1) = -\frac{3}{8}$

$= 1 + \frac{3}{2}(x-1) + \frac{3}{2!\cdot 4}(x-1)^2 - \frac{3}{3!\cdot 8}(x-1)^3$

$= \boxed{1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{3}{48}(x-1)^3}$

2. Use integrals to estimate the series below to within 0.001 of the actual value.

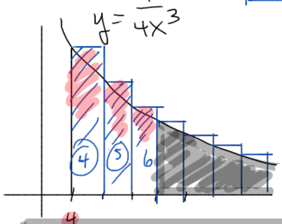
$$\sum_{n=4}^{+\infty} \frac{1}{4n^3}$$

① Find n s.t. $\frac{1}{4n^3} < 0.001$, split series, approximate the 2nd portion w/ an integral.

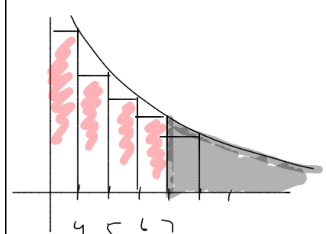
$n = 7$ works

the Series = Area of Rectangles

$\boxed{\frac{1}{4\cdot 7^3} < .001}$



$\boxed{\frac{1}{4\cdot(4)^3} + \frac{1}{4(5)^3} + \frac{1}{4(6)^3} + \int_7^{+\infty} \frac{1}{4x^3}dx} \le \sum_{n=4}^{\infty} \frac{1}{4(n)^3}$



recopy ↑ $+ \frac{1}{4\cdot 7^3} = \sum_{n=4}^{\infty} \frac{1}{4n^3} \le \frac{1}{4\cdot 4^3} + \frac{1}{4\cdot 5^3} + \frac{1}{4\cdot 6^3} + \frac{1}{4\cdot 7^3} + \int_7^{+\infty} \frac{1}{4x^3}dx$

$\frac{1}{4}\int_7^{\infty} x^{-3}dx = \frac{-1}{2}\frac{1}{4}x^{-2} = -\frac{1}{8}x^{-2}\Big|_7^{\infty}$

$= -\frac{1}{8}\left(\frac{1}{\infty^2} - \frac{1}{7^2}\right) = -\frac{1}{8}\left(-\frac{1}{49}\right)$

$= \frac{1}{392}$

Ans. is inside

$(.00941, .0103)$

$\vdash .001 \dashv$

3. The Maclaurin series for $\cos x$ is below.

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Find the interval of convergence.

$\downarrow$ increment

$(2(n+1))!$

$(2n+2)! = (2n+2)(2n+1)(2n)!$

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| \xrightarrow[n\to\infty]{}$

abs value $\Rightarrow$ ignore $(-1)^n$

$$\left|\frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}}\right| = \left|\frac{x^{2n+2}\,(2n)!}{(2n+2)!\;x^{2n}}\right| = \left|\frac{\cancel{x^{2n}} \cdot x^2 \cdot \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{x^{2n}}}\right|$$

$$= \left|\frac{x^2}{(2n+2)(2n+1)}\right| \xrightarrow[n\to\infty]{} \left|\frac{x^2}{\infty}\right| \to 0$$

$\Rightarrow$ Int. of Conv $= (-\infty, \infty)$

4. Use the series above to obtain an estimate for $\cos\left(\frac{1}{2}\right)$ to within 0.01 of the actual vallue.

Taylor's thm:

Bound derivatives of $f(x) = \cos(x)$

$|f^{(n)}(x)| \leq 1$

Fact about sin / cos

$R_n(x) \leq \frac{1 \cdot |x|^n}{n!}$ here, $x = \frac{1}{2}$

$R_n(\frac{1}{2}) \leq \frac{1 \cdot (\frac{1}{2})^n}{n!}$

Solve: when is this $< .01$

ALT. Series Test:

When you truncate an alternating series it's still accurate to within the next term.

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$\cos(\frac{1}{2}) = 1 - \frac{(\frac{1}{2})^2}{2!} + \frac{(\frac{1}{2})^4}{4!} - \frac{(\frac{1}{2})^6}{6!}$

this bounds the error in approx. with the previous terms

$$\frac{\left(\frac{1}{2}\right)^n}{n!} = \frac{1}{2^n \cdot n!} < \frac{1}{100}$$

$n = 1$
$n = 2$
$n = 3$
$n = 4 \longrightarrow \frac{1}{16 \cdot 4!} = \frac{1}{16 \cdot 24} < \frac{1}{100}$

$$\cos(1/2) \approx 1 - \frac{(1/2)^2}{2!} + \frac{(1/2)^4}{4!}.$$

5. Use an eighth degree Taylor polynomial to estimate

$$\int_0^1 \frac{\cos(x^2)-1}{x}\,dx =$$

use previous!

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

valid for all x

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!}$$

$$= \int_0^1 \frac{1 - \frac{x^4}{2!} + \frac{x^8}{4!} - 1}{x}\,dx = \int_0^1 -\frac{x^3}{2!} + \frac{x^7}{4!}$$

$$= -\frac{x^4}{8} + \frac{x^8}{8\cdot 24}\Bigg|_0^1$$

$$= -\frac{1}{8} + \frac{1}{8\cdot 24}$$

6. The Maclaruin series for the funciton $\ln(x+1)$ is below. Find the interval of convergence.

$$\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \boxed{\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n} \Rightarrow \boxed{(-1, 1]}$$

Ratio Test

$$\left| \frac{x^{n+1}}{(n+1)} \cdot \frac{n}{x^n} \right| = \left| \frac{x \cdot n}{n+1} \right| \xrightarrow[n\to\infty]{} |x|$$

Converge $\underline{\underline{\text{If}}}$ $|x| < 1 \Rightarrow (-1, 1)$

Endpoints:

$x = -1 \longrightarrow \frac{(-1)^{n+1} \cdot (-1)^n}{n} = \frac{(-1)^n \cdot (-1) \cdot (-1)^n}{n} \cdot \frac{(-1) \cdot (-1)}{n} = \frac{-1}{n}$

$\overset{1}{\underset{2n}{/\!/}}$

$\longrightarrow \sum \frac{-1}{n}$ diverges

$x = 1 \longrightarrow \sum (-1)^n \frac{1}{n}$ conv by A.S.T

7. Use the expansion above to find a series that converges to $\ln 2$.

sub: $x = 1$

8. Prove the following (most beautiful) equation is true:

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