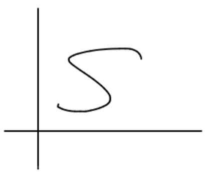
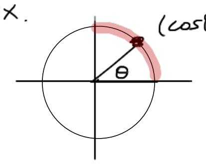
Thursday – Week 11

Parametric Curves, & arc length & area.



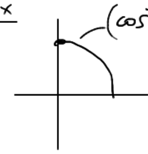
Ex.

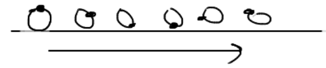


$(\cos\theta, \sin\theta)$

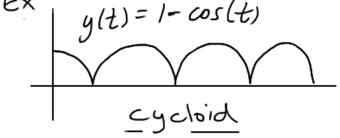
Ex

$\vec{x}(t) = (\cos t, \sin t) \quad 0 \le t \le \frac{\pi}{2}$





Ex
$x(t) = t - \sin(t)$
$y(t) = 1 - \cos(t)$

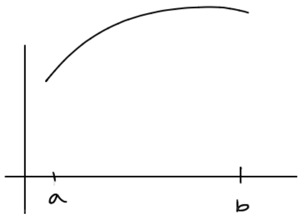


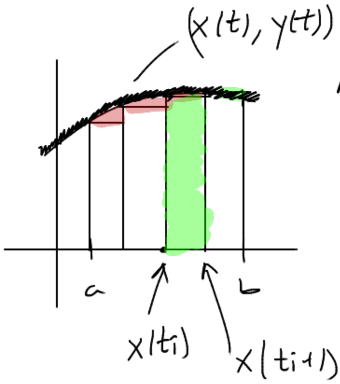
cycloid

Recall: arc length of a function of x: $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$

area under curve of function of x. $\int_a^b f(x)\, dx$

we can update these accomodate parametric curves





Approx Area = Sum of Rectangles

$$= \sum h_i w_i = \sum y(t_i)\cdot(x_{t_{i+1}} - x_{t_i}) =$$

height of each rect

width of each rect

$$\frac{\Delta x_i}{\Delta t_i} \rightarrow \frac{dx}{dt} = x'(t)$$

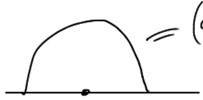
$$\Delta t_i \longrightarrow dt$$

$$= \sum_{i=0}^{n} y(t_i)\cdot \overbrace{\frac{(x_{t_{i+1}} - x_{t_i})}{t_{i+1} - t_i}}^{\frac{\Delta x_i}{\Delta t_i}} \cdot \overbrace{(t_{i+1} - t_i)}^{\Delta t_i}$$

take limit $n \to \infty$

$$= \int_a^b y(t)\cdot x'(t)\,dt = \text{Area}$$

Ex Half of unit circle



$= (\cos(t), \sin(t)) \quad 0 \le t \le \pi$

$\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

Half-angle formula

comput area under curve.

$$A = \int_a^b \underset{\underset{\sin(t)}{\shortparallel}}{y(t)} \cdot \underset{\underset{(-\sin(t))}{\shortparallel}}{x'(t)}\, dt = \int_\pi^0 -\sin^2(t)\, dt = -\int_\pi^0 \sin^2(t)\, dt$$

$$= -\int_0^\pi \left(\frac{1-\cos(2t)}{2}\right) dt = -\int_\pi^0 \frac{1}{2}\, dt + \frac{1}{2}\int_\pi^0 \cos(2t)\, dt$$

$u = 2t,\ du = 2\,dt$

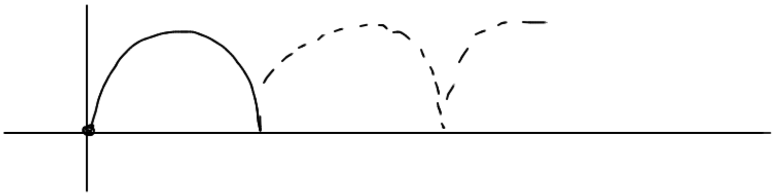
$x = 0 \Rightarrow u = 0$

$x = \pi \Rightarrow u = 2\pi$

$$= -\frac{1}{2}t\Big|_\pi^0 + \frac{1}{2}\cdot\frac{1}{2}\int_\pi^0 \cos(u)\, du$$

$$= \frac{\pi}{2} + \underbrace{\frac{1}{4}\left(-\sin(u)\right)\Big|}_{=0} = \frac{\pi}{2}$$

Next, compute area under one portion of cycloid curves.



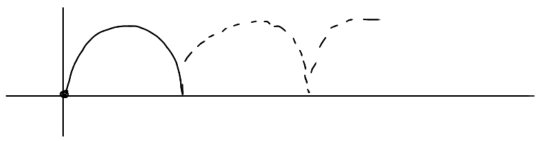
Ex $x(t) = t - \sin(t)$ $\quad 0 \le t \le 2\pi$

$y(t) = 1 - \cos(t)$

$$A = \int_0^{2\pi} y(t) \cdot x'(t)\,dt = \int_0^{2\pi} (1-\cos(t))(1-\cos(t))\,dt$$

$$= \int_0^{2\pi} 1 - 2\cos(t) + \cos^2(t$$

Next, compute area under one portion of cycloid curves.



$$\cos^2(t) = \frac{1+\cos(2t)}{2}$$

$u = 2t$
$du = 2$

$\frac{1}{4}\int \cos(u)\,du \quad = \int \frac{1}{2}\cos(2t)\,dt$

$\frac{1}{2} + \frac{1}{2}\cos(2t)$

Ex $x(t) = t - \sin(t)$ $\quad 0 \le t \le 2\pi$
$y(t) = 1 - \cos(t)$

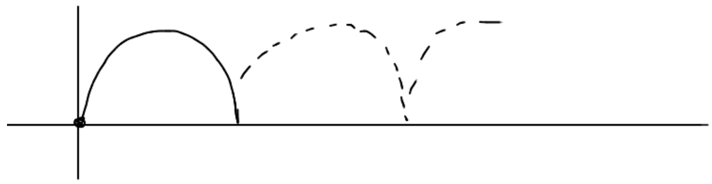
$$A = \int_0^{2\pi} y(t)\cdot x'(t)\,dt = \int_0^{2\pi} (1-\cos(t))(1-\cos(t))\,dt$$

$$= \int_0^{2\pi} 1 - 2\cos(t) + \underbrace{\cos^2(t)}\,dt = \int_0^{2\pi} 1 - 2\cos(t) + \overbrace{\frac{1+\cos(2t)}{2}}\,dt$$

$$= t - 2\sin(t) + \frac{1}{2}t + \frac{1}{4}\sin(2t)\Big|_0^{2\pi} = 2\pi - \underbrace{2\sin(2\pi)}_{=0} + \pi + \underbrace{\frac{1}{4}\sin(4\pi)}_{=0} - \underbrace{\left[0 - 2\sin 0 + \frac{1}{2}0 + \frac{1}{4}\sin 0\right]}_{=0}$$

$$= 3\pi$$

Next, compute ~~area under~~ arc length of one portion of cycloid curves.



Ex $x(t) = t - \sin(t)$ $\quad 0 \le t \le 2\pi$

$y(t) = 1 - \cos(t)$