Find the differential equaiton that satisfies the given initial condition.

(a)

$$xy'+y=y^2, y(1)=-1 \rightarrow x\frac{dy}{dx}=y^2-y \rightarrow \frac{1}{y^2-y}\,dy=\frac{1}{x}\,dx$$

$$\int \frac{1}{y^2-y}\,dy=\int \frac{1}{x}\,dx$$

Use partial fraction decomposition on the left to get to . . .

$$\int \frac{1}{y-1}-\frac{1}{y}\,dy=\frac{1}{x}\,dx$$

$$\ln(y-1)-\ln y=\ln x+C$$

$$e^{\ln(y-1)-\ln y}=e^{\ln x+C}$$

$$e^{\ln(y-1)}e^{-\ln y}=e^{\ln x}e^{C}$$

$$\frac{y-1}{y}=Ax \longrightarrow 1-\frac{1}{y}=Ax \longrightarrow \frac{1}{y}=1-Ax$$

$$y=\frac{1}{1-Ax}$$

Now use the initial condition $y(1)=-1$

$$-1=\frac{1}{1-A} \longrightarrow A=2$$

Answer:

$$y=\frac{1}{1-2x}$$

(b)

$$\frac{dy}{dt} = \frac{te^t}{y\sqrt{1+y^2}}, y(0) = 0$$

$$\int y\sqrt{1+y^2}\,dy = \int te^t\,dt$$

$$\frac{1}{3}(1+y^2)^{3/2} = te^t - e^t + C$$

Use initial value to get C:

$$\frac{1}{3}(1+0^2)^{3/2} = 0e^0 - e^0 + C \rightarrow \frac{1}{3} = -1 + C \rightarrow \frac{4}{3} = C$$

Answer:

$$\frac{1}{3}(1+y^2)^{3/2} = te^t - e^t + \frac{4}{3}$$

(c)

$$2xy' + y = 6x, x > 0, y(4) = 20$$

$$\frac{1}{2x}y + y' = 3 \longrightarrow \text{I.F.}e^{\int \frac{1}{2x}} = e^{\frac{1}{2}\ln x} = e^{\ln(x^{1/2})} = \sqrt{x}$$

$$\sqrt{x}\left(\frac{1}{2x}y + y'\right) = \sqrt{x}(3)$$

$$\frac{1}{2x^{1/2}} + x^{1/2}y' = 3x^{1/2}$$

$$\frac{d}{dx}\left(x^{1/2}y\right) = 3x^{1/2}$$

$$\int \frac{d}{dx}\left(x^{1/2}y\right)\,dx = \int 3x^{1/2}\,dx$$

$$x^{1/2}y = 2x^{3/2} + C$$

$$y = 2x + \frac{C}{\sqrt{x}}$$

$$y(4) = 20 \longrightarrow 20 = 2(4) + \frac{C}{\sqrt{4}} \longrightarrow C = 24$$

Answer:

$$y = 2x + \frac{24}{\sqrt{x}}$$

(d)

$$(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0, y(0) = 2$$

$$y' + \frac{3x}{x^2+1}y = \frac{3x}{x^2+1}$$

$$\text{I.F. } = e^{\int \frac{3x}{x^2+1}\,dx} = e^{\frac{3}{2}\ln(x^2+1)} = e^{ln\left((s^2+1)\right)^{3/2}} = (x^2+1)^{3/2}$$

$$(x^2+1)^{3/2}\left(y' + \frac{3x}{x^2+1}y\right) = \left(\frac{3x}{x^2+1}\right)(x^2+1)^{3/2}$$

$$(x^2+1)^{3/2}y' + 3x(x^2+1)^{1/2}y = 3x(x^2+1)^{1/2}$$

$$\frac{d}{dx}\left((x^2+1)^{3/2}y\right) = 3x(x^2+1)^{1/2}$$

$$\int \frac{d}{dx}\left((x^2+1)^{3/2}y\right)\,dx = \int 3x(x^2+1)^{1/2}\,dx$$

$$(x^2+1)^{3/2}y = (x^2+1)^{3/2} + C$$

$$y = 1 + \frac{C}{(x^2+1)^{3/2}}$$

$$y(0) = 2 \longrightarrow 2 = 1 + \frac{C}{(0^2+1)^{3/2}} \longrightarrow C = 1$$

Answer:

$$y = 1 + \frac{1}{(x^2+1)^{3/2}}$$