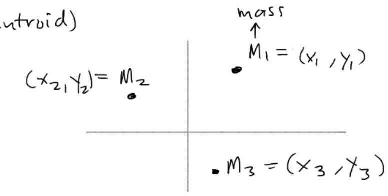
Centers of Mass (centroid)

For discrete mass



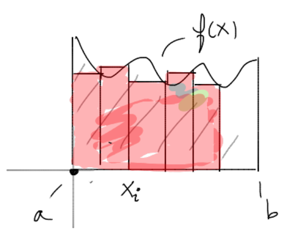
To Find the centroid

1. $X = M_1 \cdot x_1 + M_2 x_2 + M_3 x_3$
2. $Y = M_1 y_1 + M_2 y_2 + M_3 y_3$
3. $M = M_1 + M_2 + M_3$

$$\text{Centeroid} = \left(\frac{X}{M}, \frac{Y}{M} \right)$$

For masses that are not discrete, we use calculus

Lamina = Thin surface, with uniform density ρ.



Centroid:

weighted avg

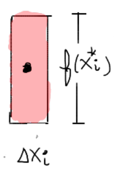
1. Moment w.r.t. x-axis $\leftrightarrow$ Y (M_x)
2. Moment w.r.t y-axis $\longleftrightarrow$ X (M_y)
3. Total Mass = M

$$\left(\frac{X}{M}, \frac{Y}{M} \right) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

Total Mass = Area × Density

$$= \int_a^b f(x)\,dx \times \rho = \boxed{\rho \int_a^b f(x)\,dx = M}$$

M_x:



Assume Mass is centered at center of rectangle — This doesn't change the moment.

Moment of Rectangle = Mass of Rectangle × Distance to x-axis

$$= \rho \cdot \text{length} \times \text{width} \times \tfrac{1}{2} f(x_i^*)$$

Single Rectangle

$$= \rho \cdot f(x_i^*) \cdot \Delta x_i \cdot \tfrac{1}{2} f(x_i^*)$$

Approx Centeroid

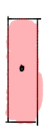
$$\sum_{i=0}^{N} \rho \frac{(f(x_i^*))^2}{2} \Delta x_i \xrightarrow[\substack{N \to \infty \\ \Delta x_i \to dx}]{\text{take lim}} \boxed{M_x = \rho \int_a^b \frac{(f(x))^2}{2}\,dx}$$

y-value

small change in x_i

infinitesimal change in x
differential of x

For M_y:



Rectangle Mass × Distance to y-axis

$$\| \quad \rho f(x_i^*) \Delta x_i \cdot x_i^*$$

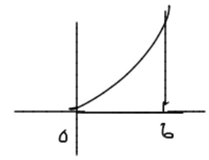
Total Approx $\rho \sum_{i=0}^{N} f(x_i^*)\, x_i^* \,\Delta x_i \xrightarrow[N \to \infty]{\lim} \boxed{\rho \int_a^b x f(x)\,dx = M_y}$

x-coord

$$\text{Centroid: } \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

$\underline{\text{Ex}}$ Find the centroid of the region bounded by:

$$y = 4x^2 + 9x \quad , \; y = 0, \; x = 0 \; , \; x = 6$$



$$M_x, M_y, M \implies \left(\frac{M_y}{M}, \frac{M_x}{M} \right) \qquad (\text{assume } \rho = 1)$$

$$M = \int_0^6 f(x)\,dx = \int_0^6 4x^2 + 9x\,dx = 450$$

$$M_x = \int_0^6 \frac{(f(x))^2}{2}\,dx = \int_0^6 \frac{(4x^2+9x)^2}{2}\,dx = 27021.6$$

$$\left(\frac{1944}{450}, \frac{27021.6}{450} \right)$$

$$M_y = \int_0^6 x \cdot f(x)\,dx = \int_0^6 x(4x^2 + 9x)\,dx = 1944$$