Linear 1^{st} order D.E.

$(ax_1 + bx_2 = c) \xrightarrow{\div a} x_1 + \frac{b}{a}x_2 = \frac{c}{a}$

linear eq'n — linear eqn

y' y

☆

$$\boxed{y' + p(x)y = q(x)}$$

Here's the method of how these are solved.

① observe LHS $\approx \frac{d}{dx}(\text{product})$, eg $\frac{d}{dx}(F \cdot G) = \underline{F'G + FG'}$

(almost)

☆'

② If we multiply eqn by $u(x)$: $u(x)\left[y' + p(x)y\right] = u(x)q(x)$

distribute

$$\underbrace{u(x) \cdot y' + u(x)p(x)\,y}_{\frac{d}{dx}\left[u(x)\cdot y\right]}$$

☆''

$\frac{d}{dx}\left[u(x)\cdot y\right] = u(x)q(x)$

$\int \frac{d}{dx}\left[u(x)\cdot y\right] = \int u(x)q(x)$

$u(x)y = \int u(x)q(x)$

$$\boxed{y = \frac{1}{u(x)}\int u(x)q(x)\,dx}$$

③ If $(u(x))' = u(x)p(x)$ then

separable

$\int \frac{du}{u}$ $\quad \frac{u'(x)}{u(x)} = p(x)$

$\ln|u(x)| = \int \frac{u'(x)}{u(x)}dx = \int P(x)dx \Rightarrow$ isolate $u(x)$ $\quad e^{\ln|u(x)|} = e^{\int P(x)dx} = e^{\int P(x)dx + C} = Ae^{\int P(x)dx}$

$e^{\ln|u(x)|} = |u(x)|$

choose $A=1$

☆

$$\Rightarrow \boxed{u(x) = e^{\int P(x)dx}}$$

integrating factor

Ex $y' + 4y = 8$

⓪ Recognize: not separable $y' = 8 \overset{\text{not product}}{-} 4y$

linear:

if $y =$ position (t)

$y' =$ velocity

$y' = 8 - 4y$

① $e^{\int P(x)dx}$ ↳ coefficient of y Integrating Factor

$$e^{\int 4dx} = e^{4x}$$

② Multiply OG. by I.F.

$$\underbrace{e^{4x} \cdot y' + e^{4x} \cdot 4y}_{\frac{d}{dx}(\text{I.F.} \cdot y)} = e^{4x} \cdot 8$$

$$\frac{d}{dx}(e^{4x} \cdot y) = 8e^{4x}$$

③ $u = 4x$, $du = 4dx$

$$\int \frac{d}{dx}(e^{4x} \cdot y) = \underbrace{\frac{1}{4}\int 8e^{4x} 4dx}_{\frac{1}{4} \cdot 8 \cdot e^{4x} = 2e^{4x}}$$

$$e^{4x}y = 2e^{4x} + C$$

④ $$y = \frac{2e^{4x} + C}{e^{4x}} = 2 + \frac{C}{e^{4x}}$$

$$\boxed{y = 2 + \frac{C}{e^{4x}}}$$ gen'l sol

Solve the I.V.P.

$$xy' = -4y + 8x \qquad y(1) = 2$$

not exactly

I.F.

$e^{\int P(x)dx}$

coef of y. when y' is "alone"

$$y' + P(x)y = q(x)$$

$$y' = -\frac{4}{x}y + 8$$

① $\boxed{y' + \frac{4}{x}y = 8}$ now in right form: $P(x) = \frac{4}{x}$

② Int. Factor: $e^{\int \frac{4}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4$

③ $x^4y' + \frac{x^4 \cdot 4y}{x} = 8x^4 \implies \underbrace{x^4y' + 4x^3y}_{} = 8x^4$

$$\frac{d}{dx}(x^4 \cdot y) = 8x^4$$

④ integrate $x^4y = \int 8x^4dx = \frac{8x^5}{5} + C$

⑤ Isolate

$$y = \frac{1}{x^4}\left[\frac{8x^5}{5} + C\right]$$

$$\boxed{y = \frac{8x}{5} + \frac{C}{x^4}}$$ gen'l sol'n

⑥ $y(1) = \frac{8}{5} + \frac{C}{1} \Rightarrow C = 2 - \frac{8}{5} = \frac{2}{5}$

$y(1) = 2$

$$\boxed{y = \frac{8x}{5} + \frac{2}{5x^4}}$$ particular sol'n

$$\int_0^1 3\tan^{-1}(x^2)$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$3\tan^{-1}(x^2) = 3\left[x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7}\right]$$

$$\|$$

$$3\int_0^1 \tan^{-1}(x^2)$$

$$= 3\int_0^1\left[x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7}\right]dx = 3\left[\frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105}\right|_0^1$$

10^{-3}

$$= 3\left(\frac{1}{3} - \frac{1}{21} + \frac{1}{55} - \frac{1}{105} + \frac{1}{171}\right)$$

$$\boxed{\text{10.6–10.8 Part \#4 \quad \#5}}$$