First-Order Differential Equations

Separable	Linear	Neither
$y' = f(x)g(y)$	$y' + P(x)y = Q(x)$	Example: $y' = y^2 + x$

Newton's Law of Cooling

$$\frac{dT}{dt} = k(A - T)$$

ambient

temp of material

proportionality constant

Sol'n to a diff. eqn is a function

$x^2 = 9$

$x = \pm 3$

$$\frac{dT}{A-T} = k\,dt$$

say, $A = 40°$

$$\int \frac{dT}{40-T} = \int k\,dt$$

$$\ln|40-T| = -kt + C$$

$$|40-T| = e^{-kt+C} = e^{-kt} e^{C} = Ae^{-kt} \Rightarrow \boxed{T = 40 - Ae^{-kt}} = \boxed{40 + Be^{-kt}}$$

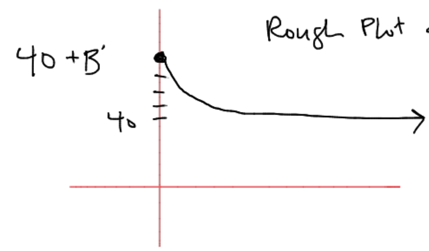
$e^C = A$

drop $|\cdot|$ b/c $A = \pm$

If $A < T \Rightarrow A < 0$

$B > 0 = T_e$

Rough Plot of Coffee Temp



The second (and last) kind of D.E. you'll see in MA163 is

First-order Linear → looks like $ax_1 + bx_2 = c$

1st derivative only

↵ you could ÷ a

$x_1 + \frac{b}{a}x_2 = \frac{c}{a}$

$$\boxed{y' + p(x)y = q(x)}$$

$(F\cdot G)' = \underbrace{F'G + F\cdot G'}$

1. Note LHS "looks like (almost)" a derivative of a product:

2. Multiply entire eq'n by $u(x)$: $u(x)[y' + p(x)y] = u(x)q(x)$

$$\underbrace{u(x)\cdot y' + u(x)p(x)\cdot y} = u(x)q(x)$$

3. This works if $u'(x) = u(x)p(x)$. (separable) $\Rightarrow \int \frac{u'}{u(x)}\,dx = \int p(x)\,dx$

$$\ln|u(x)| = \int p(x)\,dx$$

4. Hit w/ e: $|u(x)| = e^{\int p(x)dx} = e^{\int p(x)dx + C} = e^{\int p(x)dx}\cdot e^{C}$ set $=1 \Rightarrow$ drop abs val.

$$\boxed{u(x) = e^{\int p(x)dx}}$$ integrating factor

5. So... when given $\boxed{y' + p(x)y = q(x)}$ (goal: find function y satisfies this)

do (i) Form integrating factor $e^{\int p(x)dx}$

(ii) multiply by it

$$\underbrace{e^{\int p(x)}\left[y' + p(x)y\right]} = e^{\int p(x)}\cdot q(x)$$

$$\underbrace{y'e^{\int p(x)} + yp(x)}$$

(iii) interpret LHS as a derivative of product

$$\frac{d}{dx}\left(e^{\int p(x)}\cdot y\right) = e^{\int p(x)} q(x)$$

(iv) Integrate: $\int$ cancels $\frac{d}{dx}$

$$\int \frac{d}{dx}\left(e^{\int p(x)}\cdot y\right) = \int e^{\int p(x)} q(x)$$

$$e^{\int p(x)}\cdot y = \int e^{\int p(x)} q(x)$$

(v) isolate y

$$y = \frac{1}{e^{\int p(x)}}\int e^{\int p(x)}\cdot q(x)\,dx$$

Ex:

$$y' + 4y = 8$$

① Form I.F. $\quad e^{\int 4\,dx} = e^{4x}$

② Multiply by it: $\quad e^{4x}[y' + 4y] = e^{4x} \cdot 8$

③ distribute $\quad \underbrace{e^{4x} \cdot y' + e^{4x} \cdot 4y} =$

④ LHS $= \frac{d}{dx}$ $\quad \frac{d}{dx}\left(e^{4x} \cdot y\right) \quad \frac{d}{dx}(\text{Int. Factor} \times y) =$

⑤ integrate $\quad e^{4x} y = \frac{1}{4}\int \underbrace{8e^{4x}\,4\,dx}_{\substack{u=4x \\ du=4dx}} = \frac{1}{4}\int 8e^{u}\,du = 2e^{4x} + C$

⑥ Isolate $\quad y = \frac{1}{e^{4x}}\left[2e^{4x} + C\right] = 2 + \frac{C}{e^{4x}} \overset{\text{or}}{=} 2 + Ce^{-4x}$

$$\boxed{y = 2 + Ce^{-4x}}$$

Check:

$$(2 + Ce^{-4x})' + 4 \cdot (2 + Ce^{-4x}) \overset{?}{=} 8$$

$$-4 \cdot Ce^{-4x} + 8 + 4Ce^{-4x} = 8 \checkmark$$

Solve the Initial Value Problem

$$y' + \frac{4}{x}y = 8 \quad // \quad y(1) = 2$$

(0) Recognize Linear 1st Order

(1) I.F. — any I.F. works so let $C = 0$

$$e^{\int \frac{4}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4$$

(2) Mult

$$x^4 y' + x^4\left(\frac{4}{x}\right)y = 8x^4$$

$$\underbrace{x^4 y' + 4x^3 y}_{\frac{d}{dx}(x^4 \cdot y)} = 8x^4$$

$$\boxed{y = \frac{8}{5}x + \frac{2}{5x^4}}$$

(3) Int.

$$x^4 y = \int 8x^4\,dx = \frac{8x^5}{5} + C$$

$$y = \frac{1}{x^4}\left(\frac{8x^5}{5}\right) + \frac{C}{x^4}$$

$$y = \frac{8}{5}x + \frac{C}{x^4}$$

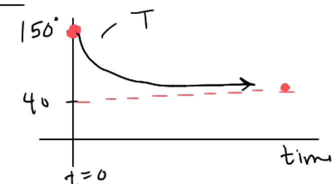
gen'l sol'n

(4) particular sol'n

$$\underset{2}{\overset{\|}{y(1)}} = \frac{8}{5}(1) + \frac{C}{1} = \frac{8}{5} + C$$

$$\Rightarrow C = 2 - \frac{8}{5} = \frac{2}{5}$$

wk 12 Fri



· Today: Linear - first order D.E's.

But First

Newton's Law of Cooling: $\frac{dT}{dt} = k(A-T)$ — separable

k → proportionality constant (depends on surface area of object)

$A = 40°$ ambient temp

$T_0 = T(0)$ initial temp of object $= 150°$

$T =$ temp of object @ time t

$\frac{dT}{dt}$ rate of cooling

Separable D.E $y' = f(x) \cdot g(y)$

$\Rightarrow \frac{dT}{A-T} = k\,dt$

$\Rightarrow \int \frac{dT}{A-T} = \int k\,dt$

$\ln|A-T| = -kt + C$

$e^{\ln|A-T|} = e^{-kt+C} = e^{-kt} \cdot e^{C} = Be^{-kt}$

$\|$

$|A-T| \Rightarrow$ drop abs val

$A - T = Be^{-kt}$

solve for T:

$$\boxed{T = A - Be^{-kt}}$$