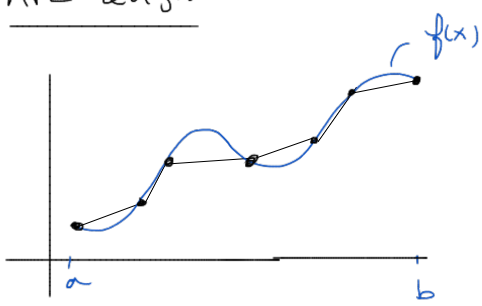
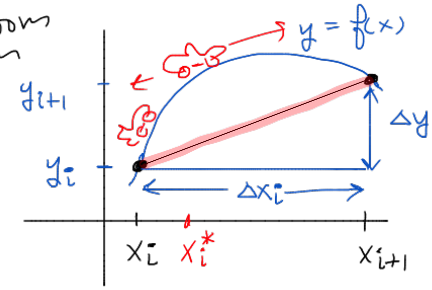
Arc length:



what's the length of this curve?

Approximate by summing length of straight lines breaking up the curve.

zoom in



dist b/w two pts

$$\text{Approx Length} = \sum_{i=0}^{N} \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}$$

$$= \sum_{i=0}^{N} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

Factor $(\Delta x_i)^2$ out

$$= \sum_{i=0}^{N} \sqrt{(\Delta x_i)^2\left(1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right)} = \sum_{i=0}^{N} \Delta x_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}$$

slope of hypotenuse above

Mean Value theorem: you can find an x-value b/w x_i & x_{i+1} whose derivative at this point (slope of tangent line) equals $\frac{\Delta y_i}{\Delta x_i}$, call it x_i^*.

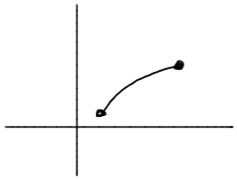
i.e., $$\frac{\Delta y_i}{\Delta x_i} = f'(x_i^*)$$

$$= \sum_{i=0}^{N} \Delta x_i \sqrt{1 + [f'(x_i^*)]^2}$$

this is the approx length, depends on N. Improve by increasing N.

$$\text{Arc Length} = \lim_{N\to\infty} \underbrace{\sum_{i=0}^{N}}_{\int} \underbrace{\Delta x_i}_{dx} \sqrt{1 + [f'(\underbrace{x_i^*}_{x})]^2} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Ex Find the length of the curve $y = \ln(\sec(x))$ $\quad 0 \le x \le \pi/4$



$$\text{Length} = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x)$$

$$= \int_0^{\pi/4} \sqrt{\underbrace{1 + \tan^2(x)}_{\sec^2(x)}}\, dx = \int_0^{\pi/4} \sec(x)\, dx$$

hit w/ $\dfrac{\sec x + \tan x}{\sec x + \tan x}$

$$= \int_0^{\pi/4} \sec(x) \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\, dx = \int_0^{\pi/4} \frac{\sec^2 x + \sec x \tan x}{\sec(x) + \tan(x)}\, dx = \int_1^{\sqrt{2}+1} \frac{du}{u}$$

$u = \sec(x) + \tan(x)$

$x = 0 \Rightarrow u = \frac{1}{\cos(0)} + \tan(0) = 1$

$x = \pi/4 \Rightarrow u = \frac{1}{\cos(\pi/4)} + \tan(\pi/4) = \sqrt{2} + 1$

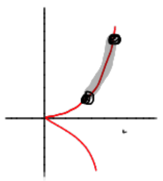
$$= \ln|u| \Big|_1^{\sqrt{2}+1} = \ln(\sqrt{2}+1) - \ln(1) = \ln(\sqrt{2}+1)$$

<u>Many</u> arc length problems produce integrals that are hard to compute

— one approximates the sol'n

— <u>computer</u>

Ex $\boxed{y^2 = x^3}$. Find the length of this curve b/w $(1,1)$ & $(4,8)$.



$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \left(\tfrac{3}{2}\sqrt{x}\right)^2}$$

$$= \int_1^4 \sqrt{1 + \tfrac{9}{4}x}\; dx = \int_{\frac{13}{4}}^{10} \sqrt{u}\,\tfrac{4}{9}\,du$$

$u = 1 + \frac{9}{4}x$ | $x=1,\ u = \frac{13}{4}$ | $\frac{4}{9}\cdot\frac{2}{3}\, u^{\frac{3}{2}}\Big|_{\frac{13}{4}}^{10}$

$du = \frac{9}{4}\,dx$ | $x=4,\ u=10$

$\frac{4}{9}du = dx$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^3\right)$$

↓ chain rule

$$2\cdot y\cdot\frac{dy}{dx} = 3x^2 \underbrace{\frac{dx}{dx}}_{=1} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

what is y? $y^2 = x^3$

$y = \pm x^{3/2}$ on our branch $y = x^{3/2}$

$$= \frac{3x^2}{2\left(x^{3/2}\right)} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2} \qquad \frac{dy}{dx} = \frac{3\sqrt{x}}{2}$$