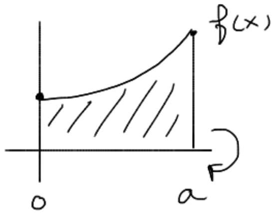
Wednesday – Week 12

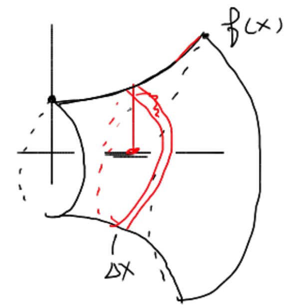
Today: surface Area / Arc length

Recall: Solids of Revolution:



revolve about $\underline{x}$ – axis $\longrightarrow$

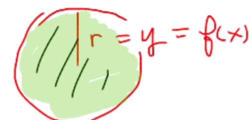
$$\text{Vol} = \int_0^a \pi (f(x))^2 dx$$



Surface Area:

$$S = \int_0^a 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\boxed{S = \int_0^a 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$



sub

$$\text{Vol of Disk} = \pi r^2 \cdot \Delta x$$

$$V = 2\pi f(x) \Delta x$$

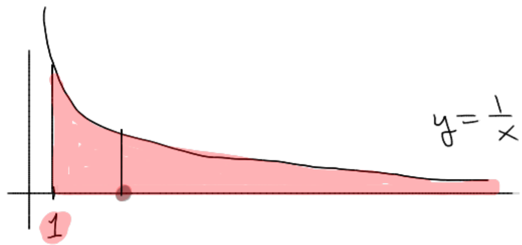
$$\text{Approx Vol} = \sum_{i=1}^{N} \pi (f(x))^2 \Delta x_i$$

$$\text{Actual Vol of Solid} = \lim_{N \to \infty} \sum_{i=1}^{N} \pi f^2(x) \Delta x_i$$

$$= \int_0^a \pi f^2(x)\, dx$$

Gabriel's Horn & the Painter's Paradox

$f(x) = \frac{1}{x}$



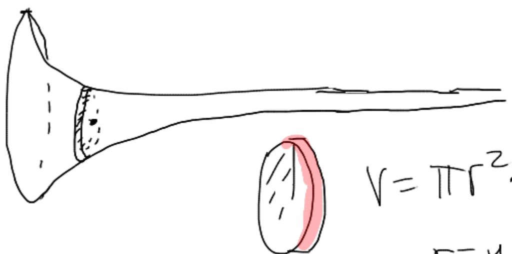
revolve about x-axis

Already we know:

$$V = \int_1^\infty \pi\left(\frac{1}{x^2}\right)dx = \pi\left(\frac{-1}{x}\right)\Big|_1^\infty$$

$$= \pi\left(\underset{\overset{\shortparallel}{0}}{\frac{-1}{\infty}} - \left(\frac{-1}{1}\right)\right)$$

$$= \pi$$

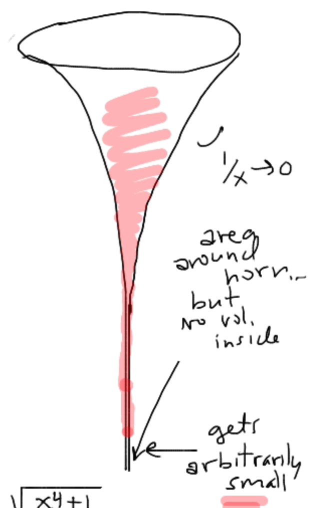


$V = \pi r^2 \cdot \Delta x = \pi\left(\frac{1}{x}\right)^2 \Delta x$

$r = y = \frac{1}{x}$

Finite Volume

$\boxed{\text{So only a finite amount of paint fits inside the horn}}$



Now Surface Area

$$S = \int_1^\infty \underbrace{2\pi y}_{\text{circ.}} \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}_{\text{arc length}} = \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx$$

claim: S is infinite.

$$= 2\pi\int_1^\infty \frac{1}{x}\sqrt{\frac{x^4}{x^4} + \frac{1}{x^4}}\, dx$$

$$= 2\pi\int_1^\infty \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\, dx = 2\pi\int_1^\infty \frac{1}{x}\frac{\sqrt{x^4+1}}{x^2}\, dx$$

$\boxed{\text{An infinite amount of paint required to paint one side.}}$

$$= \underbrace{2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx}_{\boxed{\text{Surface area}}} \underset{\text{Fact}}{>} 2\pi\int_1^\infty \frac{\overset{= x^2}{\sqrt{x^4}}}{x^3}\, dx$$

$$= 2\pi\int_1^\infty \frac{1}{x}\, dx$$

$$= 2\pi \cdot \ln|x|\Big|_1^\infty$$

$$= 2\pi\left(\ln(\infty) - \overset{\overset{0}{\shortparallel}}{\ln(1)}\right)$$

$$= \infty$$

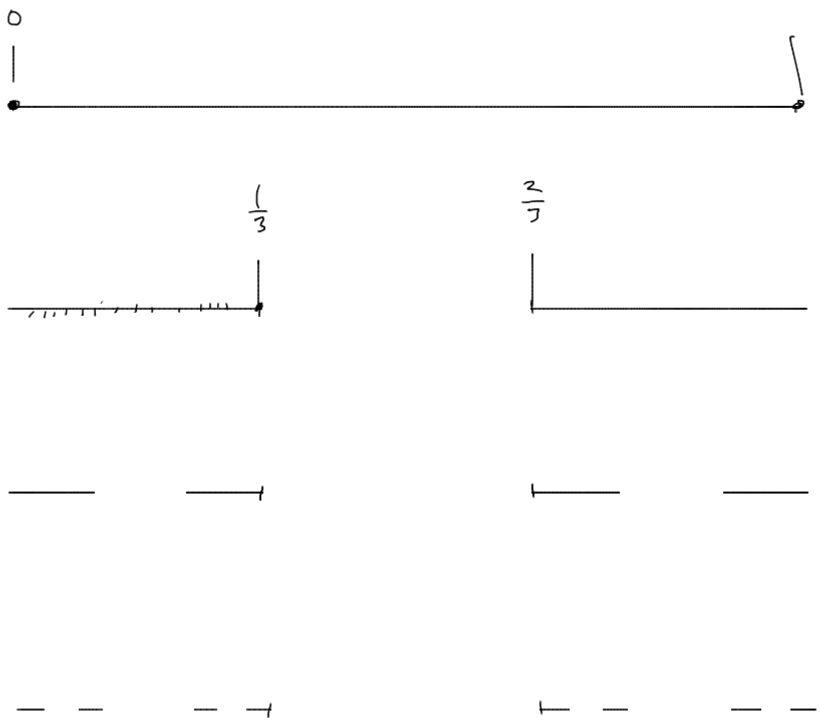
P. Paradox: How do we paint an infinite canvas w/ finite paint? or don't we?

Any Finite amount of paint contains a finite # of molecules $\Rightarrow$ $\boxed{\text{there's one with smallest size}}$

Set theory: Any finite list of positive #'s has a smallest member.

this relates to:

<u>Cantor Set</u>



length

- start w/ $[0,1]$ (length 1) — 1
- remove the middle $\frac{1}{3}$ — $1-\frac{1}{3}$
- remove the middle $\frac{1}{3}$ of each — $1-\frac{1}{3}-2\left(\frac{1}{3}\right)^2$
- remove the middle $\frac{1}{3}$ of each — $1-\frac{1}{3}-2\left(\frac{1}{9}\right)-4\left(\frac{1}{27}\right)$

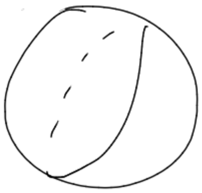
continue
⋮

$\xrightarrow{\text{limit}} \emptyset$

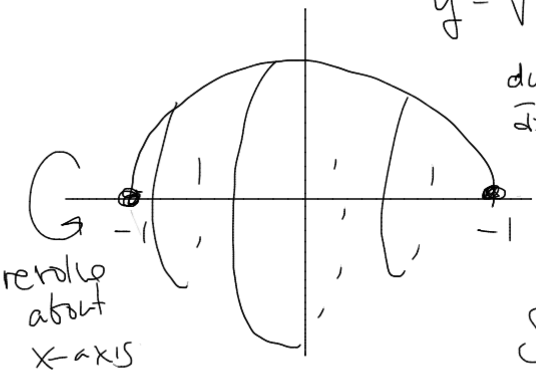
Remove all length
get points (dust)
remain

See:
Sierpinski Gasket

Surface Area of a Sphere:



$$S = 4\pi r^2 \qquad \text{why?}$$



$$y = \sqrt{1-x^2}$$

(top $\frac{1}{2}$ of a circle w/ radius = 1)

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{(1-x^2)^{1/2}}$$

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \frac{1-x^2}{1-x^2}$$

$$= \int_{-1}^{1} 2\pi \sqrt{1-x^2}\sqrt{1 + \left(\frac{x^2}{1-x^2}\right)}\, dx$$

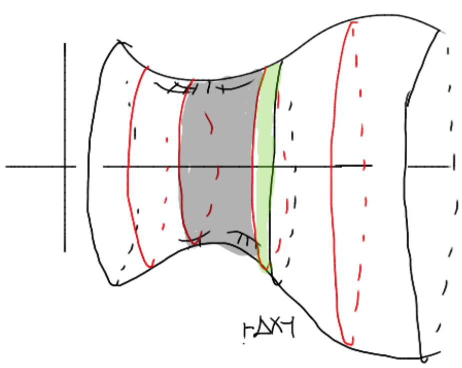
$$= \int_{-1}^{1} 2\pi \sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

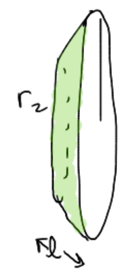
$$= \int_{-1}^{1} 2\pi \cdot 1\, dx = 2\pi x \Big|_{-1}^{1} = 2\pi - (2\pi(-1))$$

$$= 4\pi$$

Wed. - Week 12

Surface area of solids of revolution





, Area of this band

$A = 2\pi r \ell$

$r = \frac{1}{2}(r_1 + r_2)$

(average)

Fact: when Δx is small

$r_i \approx f(x_i^*)$ & $\ell \approx$ arc length $\sqrt{1+(f'(x_i))^2}$

avg radius

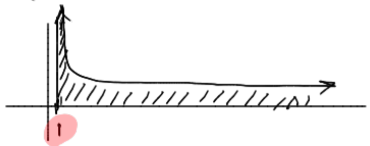
Approx Surface Area $\sum_{i=1}^{N} 2\pi r_i \ell_i$

$$\boxed{\text{Actual Surface Area} = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx}$$

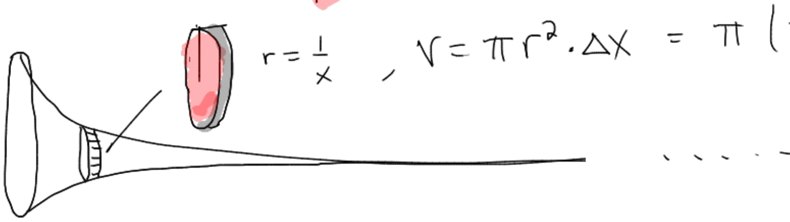
when revolve about x-axis

Gabriel's Horn & the Painters Paradox

$f(x) = \frac{1}{x}$



revolve about x-axis

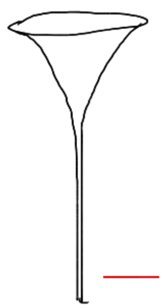


$r = \frac{1}{x}$, $V = \pi r^2 \cdot \Delta x = \pi\left(\frac{1}{x}\right)^2 \Delta x$

$$Vol = \int_1^{\infty} \pi\left(\frac{1}{x}\right)^2 dx = \pi\left(\frac{-1}{x}\right)\Bigg|_1^{\infty} = \pi\left(\frac{-1}{\infty} - \left(\frac{-1}{1}\right)\right) = \pi$$

has a smallest molecule

Finite Volume!



Finite amount of paint will fill entire Horn

despite the horn being infinitely long

Surface Area:

$$\int_1^{\infty} 2\pi \frac{1}{x}\sqrt{1+(f'(x))^2}\, dx = \int_1^{\infty} 2\pi \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\, dx$$

$$= 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx = 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\, dx \overset{\text{Fact}}{>} 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{\frac{x^4}{x^4}}\, dx$$

$$= \underbrace{2\pi\int_1^{\infty} \frac{1}{x}\, dx}$$

$$= 2\pi \ln|x| \Big|_1^{\infty}$$

$$= 2\pi \cdot \ln(\infty) - 2\pi \cdot \ln(1)$$

$$= \infty$$

The horn has infinite surface area

Example of using the surface area formula

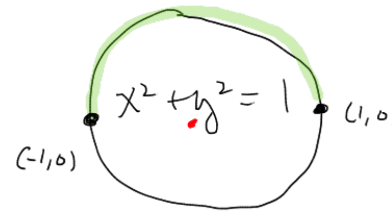
$S = 4\pi r^2$

For $r = 1$, $\therefore S = 4\pi$

$\frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$

$y' = \frac{-x}{\sqrt{1-x^2}}$ $\Big|\Big|$ $(y')^2 = \frac{x^2}{1-x^2}$

Top Half of a circle w/ radius 1 $= y = \sqrt{1-x^2}$ b/c



revolve about x-axis, sweep out a sphere w/ $r = 1$

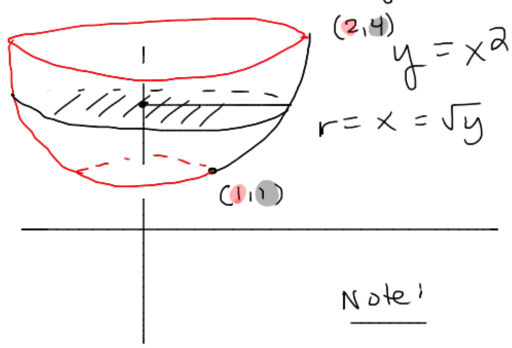
Surface Area: $\int_{-1}^{1} 2\pi\sqrt{1-x^2} \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-1}^{1} 2\pi\sqrt{1-x^2} \cdot \sqrt{1 + \frac{x^2}{1-x^2}}$

$= \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}} = \int_{-1}^{1} 2\pi\sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}}\, dx = \int_{-1}^{1} 2\pi\, dx$

$= 2\pi \int_{-1}^{1} 1\, dx$

$= 2\pi x \Big|_{-1}^{1} = 4\pi$

Revolve about y-axis:



note: x, not x^2 b/c x = radius

$$\int_1^2 2\pi \cdot x \cdot \sqrt{1+(2x)^2}\,dx$$

Note:

Int. wrt y:

$$\int_1^4 2\pi \cdot \sqrt{y} \cdot \sqrt{1+\left(\tfrac{1}{2}y^{-1/2}\right)^2}\,dy$$

$$\frac{dx}{dy} = (\sqrt{y})' = \frac{1}{2}y^{-1/2}$$