Friday - Week 13

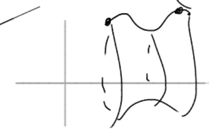
Two items that will help take-home

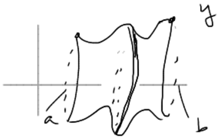
- Area of Parametric Surface
- Integration detail for arc length problem

Area of Parametric Surfaces of Revolution

what we already know ; $y = f(x)$

$(x(t), y(t))$

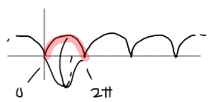




$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

$$A = \int_{t_0}^{t_1} 2\pi \cdot y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Example: $x(t) = t - \sin t$
$y(t) = 1 - \cos t$



Area

$$A = \int_0^{2\pi} 2\pi \cdot (1 - \cos t)\sqrt{(1-\cos t)^2 + (\sin t)^2}\,dt$$

$\sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{1}}$

$\sqrt{2 - 2\cos t} = \sqrt{2(1 - \cos t)}$

$\sqrt{2}\sqrt{1 - \cos t}$

$$= 2\pi\sqrt{2}\int_0^{2\pi}(1 - \cos t)^{3/2}\,dt$$

$2\sin^2 t = \frac{1 - \cos(2t)}{2}$

or

$2\sin^2(t/2) = 1 - \cos(t)$

$$= 2\pi\sqrt{2}\int_0^{2\pi}\left(2\sin^2(t/2)\right)^{\frac{3}{2}}dt$$

$(A^2)^{\frac{3}{2}} = (\sqrt{A^2})^3 = |A|^3$

$\sqrt{(-9)^2} = \sqrt{81} = 9 = |-9|$

$$= 2\pi\sqrt{2}\int_0^{2\pi} 2^{\frac{3}{2}} \cdot |\sin(t/2)|^3\,dt$$

$2^{3/2} = 2\cdot\sqrt{2}$

$(2\sqrt{2})(2\sqrt{2}) = 8$

$$= 8\pi\int_0^{2\pi}|\sin(t/2)|^3\,dt$$

can we drop the abs. value ?

- is $\sin(t/2) > 0$ everywhere on $[0, 2\pi]$

$$= 8\pi\int_0^{2\pi}\sin^3(t/2)\,dt$$

$\boxed{\begin{array}{l}\sin^3(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)\\ x = t/2 \ldots\\ \sin^3(t/2) = \frac{3}{4}\sin(t/2) - \frac{1}{4}\sin\left(\frac{3t}{2}\right)\end{array}}$

u-sub ...

$$= 2\pi\int_0^{2\pi} 3\sin(t/2) - \sin\left(\frac{3t}{2}\right)\,dt$$

B/C: know!
$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos x$
$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\sin 3x = \sin(2x + x) = \sin(2x)\cos x + \sin x \cdot \cos 2x$
$= 2\sin x\cos x \cdot \cos x + \sin x\,(\cos^2 x - \sin^2 x)$
$= 2\sin x\cos^2 x + \sin x\cos^2 x - \sin^3 x$
$= 3\sin x\cos^2 x - \sin^3 x$
$= 3\sin x(1 - \sin^2 x) - \sin^3 x$
$= 3\sin x - 4\sin^3 x$

$$= 2\pi\left[6\int_0^{\pi}\sin u - \frac{2}{3}\int_0^{\pi}\sin u\right]dt$$

$u = t/2$
$du = 1/2$

$$= 2\pi\left[-6\cos u\Big|_0^{\pi} + \frac{2}{3}\cos u\Big|_0^{\pi}\right] =$$

$$= 2\pi\left[-6\cdot\cos\pi + 6\cos 0\right] + \frac{2}{3}\left[\cos\pi - \cos 0\right]$$

$$= 2\pi\left(6 + 6 - \tfrac{2}{3} - \tfrac{2}{3}\right) = 2\pi\left(10\tfrac{2}{3}\right) = 2\pi\left(\frac{32}{3}\right) = \frac{64\pi}{3}$$

$21\pi + \frac{\pi}{3}$

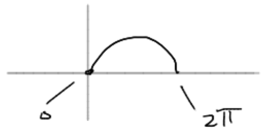
Arc length (involving abs value)

$x_1(t) = t - \sin(t)$

$y_1(t) = 1 - \cos t$

$x(t) = t - \sin t$

$y(t) = 1 - \cos t$



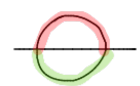
$$L = \int_0^{2\pi} \sqrt{(x_1'(t))^2 + (y_1'(t))^2}\, dt \quad \circledast \quad = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = \sqrt{2} \int_0^{2\pi} \sqrt{(1-\cos t)}\, \frac{\sqrt{1+\cos t}}{\sqrt{1+\cos t}}\, dt$$

$\sqrt{1-\cos^2(t)}$ " $\sqrt{\sin^2 t}$ " $|\sin(t)|$

$$= \sqrt{2} \int_0^{2\pi} \frac{|\sin(t)|}{\sqrt{1+\cos t}}\, dt$$

$$= \sqrt{2}\left[\int_0^{\pi} \frac{\sin(t)}{\sqrt{1+\cos t}}\, dt + \int_{\pi}^{2\pi} \frac{-\sin(t)}{\sqrt{1+\cos t}}\, dt \right]$$



$u = 1 + \cos t$

$du = -\sin t$

$$\sqrt{2}\left[\int_2^0 \frac{-du}{\sqrt{u}} + \int_0^2 \frac{du}{\sqrt{u}} \right] = \sqrt{2}\left(-2\sqrt{u}\Big|_2^0 + 2\sqrt{u}\Big|_0^2 \right)$$

$u^{-1/2} \longrightarrow 2u^{1/2}$

$$= \sqrt{2}\left(0 - (-2\sqrt{2}) + 2\sqrt{2} - 0\right)$$

$$= \sqrt{2}\left(2\sqrt{2} + 2\sqrt{2}\right) = \sqrt{2}(4\sqrt{2})$$

$$= 8 \checkmark$$